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## ABSTRACT

We present upper estimates for the number of negative eigenvalues of two-dimensional Schrödinger operators with potentials generated by Ahlfors regular measures of arbitrary fractional dimension  $\alpha \in (0, 2]$ . The estimates are given in terms of integrals of the potential with a logarithmic weight and of its  $L \log L$  type Orlicz norms. In the case  $\alpha = 1$ , our results are stronger than the known ones about Schrödinger operators with potentials supported by Lipschitz curves.

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## I. INTRODUCTION

Given a non-negative function  $V \in L^1_{\text{loc}}(\mathbb{R}^d)$ , consider the Schrödinger operator on  $L^2(\mathbb{R}^d)$ ,

$$H_V := -\Delta - V, \quad V \geq 0, \quad (1)$$

where  $\Delta := \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}$ . This operator is defined by its quadratic form

$$\begin{aligned} \mathcal{E}_{V, \mathbb{R}^d}[u] &= \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx - \int_{\mathbb{R}^d} V(x)|u(x)|^2 dx, \\ \text{Dom}(\mathcal{E}_{V, \mathbb{R}^d}) &= \left\{ u \in W_2^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, V(x)dx) \right\}. \end{aligned}$$

Denote by  $N_-(\mathcal{E}_{V, \mathbb{R}^d})$  the number of negative eigenvalues of  $H_V$  counted according to their multiplicity. An estimate for  $N_-(\mathcal{E}_{V, \mathbb{R}^d})$  in the case  $d \geq 3$  is given by the celebrated Cwikel–Lieb–Rozenblum inequality,

$$N_-(\mathcal{E}_{V, \mathbb{R}^d}) \leq C_d \int_{\mathbb{R}^d} V(x)^{d/2} dx \quad (2)$$

(see, e.g., Refs. 1–3 and the references therein). If  $V \in L^{d/2}(\mathbb{R}^d)$ , then this estimate implies that

$$N_-(\mathcal{E}_{\lambda V, \mathbb{R}^d}) = O(\lambda^{d/2}) \text{ as } \lambda \rightarrow +\infty. \quad (3)$$

The estimate is optimal in the sense that (3) implies that  $V \in L^{d/2}(\mathbb{R}^d)$  [see, e.g., Ref. 4, (127)].

It is well known that (2) does not hold for  $d = 2$ . In this case, the Schrödinger operator has at least one negative eigenvalue for any nonzero  $V \geq 0$ , and no estimate of the type

$$N_-(\mathcal{E}_{V,\mathbb{R}^2}) \leq \text{const} + \int_{\mathbb{R}^2} V(x)W(x) dx$$

can hold, provided that the weight function  $W$  is bounded in a neighborhood of at least one point (see Ref. 5). Most known upper estimates for  $N_-(\mathcal{E}_{V,\mathbb{R}^2})$  involve terms of two types: integrals of  $V$  with a logarithmic weight and  $L \log L$  type (or  $L_p, p > 1$ ) Orlicz norms of  $V$  (see Refs. 5–10 and the references therein). The following inequality is an example of such estimates:

$$N_-(\mathcal{E}_{V,\mathbb{R}^2}) \leq 1 + \text{const} \left( \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) dx + \|V\|_{\mathcal{B},\mathbb{R}^2} \right), \quad \forall V \geq 0,$$

where  $\|\cdot\|_{\mathcal{B},\mathbb{R}^2}$  denotes the Orlicz norm (8), (10). It was proved in Ref. 9, where it was also shown to be equivalent to the estimate conjectured in Ref. 11 and weaker than the one obtained in Ref. 10 (see Ref. 9 for stronger estimates). Ideally, one would like to have an optimal estimate of the type

$$N_-(\mathcal{E}_{V,\mathbb{R}^2}) \leq 1 + \Xi(V), \tag{4}$$

where  $\Xi$  is a combination of certain norms,  $\Xi(\lambda V) = O(\lambda)$  as  $\lambda \rightarrow +\infty$ , and, most importantly,

$$N_-(\mathcal{E}_{\lambda V,\mathbb{R}^2}) = O(\lambda) \text{ as } \lambda \rightarrow +\infty \tag{5}$$

implies that  $\Xi(V) < \infty$ . Unfortunately, even the strongest known estimates for  $d = 2$  are not optimal in this sense (see Ref. 9). Finding an optimal estimate of type (4) seems to be a difficult problem. The estimates for  $N_-(\mathcal{E}_{V,\mathbb{R}^2})$  with  $V$  supported by the Lipschitz curves obtained in Refs. 12 and 13 show that (5) may hold for singular potentials supported by lower-dimensional sets. We believe that a better understanding of Schrödinger operators with such singular potentials (supported by fractal sets) might shed some additional light on the above problem. This was the main motivation for the present work, although the results obtained here might be of some relevance to the study of fractal antennae, apertures, screens, and transducers (see, e.g., Refs. 14–19 and the references therein), especially in the case of impedance (Robin) boundary conditions (see Refs. 20–23).

In this paper, we deal with the operator

$$H_{V\mu} := -\Delta - V\mu, \quad V \geq 0, \tag{6}$$

on  $L^2(\mathbb{R}^2)$ , where  $V \in L^1_{\text{loc}}(\mathbb{R}^2, \mu)$  and  $\mu$  is a  $\sigma$ -finite positive Radon measure on  $\mathbb{R}^2$  that is Ahlfors regular of dimension  $\alpha \in (0, 2]$  [see (26)]. We provide a unified treatment of potentials locally integrable with respect to the Lebesgue measure on  $\mathbb{R}^2$  ( $\alpha = 2$ ), potentials supported by curves ( $\alpha = 1$ ), and potentials supported by sets of fractional dimension  $\alpha \in (0, 1) \cup (1, 2)$ . In the case  $\alpha = 2$ , we get the same estimate as in Ref. 9 (Theorem 6.1), which is stronger than most other known estimates that use isotropic norms. [Anisotropic norms like the ones used in Ref. 9 (Sec. 7) and Ref. 24 are not available in the case  $\alpha < 2$  and hence are not treated here.] In the case  $\alpha = 1$ , our Theorem 3.1 and Corollary 3.2 are stronger than the results obtained in Refs. 12 and 13 as we are now able to cover Ahlfors regular curves rather than just Lipschitz ones. In the case  $\alpha \in (0, 1) \cup (1, 2)$ , our results seem to be completely new.

The proof of our main result, Theorem 3.1, follows the same blueprint as in Refs. 10 and 9, but dealing with measures supported by sets of fractional dimension causes quite a few difficulties. Some of them are listed below.

- (1) One of the key technical ingredients in Ref. 10 (and in Ref. 9) was a result saying that the Orlicz norm of the potential over a square of the side length  $t > 0$  with a fixed center is a continuous function of  $t$ . This is no longer true for potentials of the form  $V\mu$  [see (6)] if the measure  $\mu$  is supported by an  $\alpha$ -dimensional set with  $\alpha \in (0, 1]$  and hence can charge the sides of the square. Lemma 2.13 allows one to choose the directions of the sides of the square in such a way that this difficulty is avoided (see Lemma 2.15).
- (2) The Birman–Laptev–Solomyak method (see Sec. IV) used in this paper (and in Refs. 10 and 9) splits the problem into radial and non-radial parts. The former is essentially a one-dimensional problem and is usually easier to handle than the latter. If the measure  $\mu$  is supported by an  $\alpha$ -dimensional set with  $\alpha \in (0, 2)$ , then the radial operator corresponding to (6) is a one-dimensional Schrödinger operator whose potential is a measure that may be supported by a set of a fractional dimension and may even have atoms if  $\alpha \in (0, 1]$ . Hence, one needs to extend to such operators appropriate estimates known for Schrödinger operators with potentials locally integrable with respect to the one-dimensional Lebesgue measure.<sup>25</sup> This has been carried out in Ref. 26.
- (3) The Birman–Laptev–Solomyak method allows one to obtain spectral estimates for the non-radial part of the problem mentioned above by splitting  $\mathbb{R}^2 \setminus \{0\}$  into homothetic annuli centered at 0, getting an estimate for one of those annuli, and then extending it by scaling to all other ones. Getting an estimate for an annulus usually involves covering it by carefully chosen squares, and an additional difficulty in the case of operator (6) is that one has to distinguish between squares that are centered in the support of the measure  $\mu$  and those that are not. Obviously, this complication does not arise in the standard case where  $\mu$  is the two-dimensional Lebesgue measure. Extending an estimate to all annuli by scaling is also not entirely trouble free for operator (6) as the measure  $\mu$  does not have to be homogeneous. Scaling leads to a change of measure, and one needs explicit information on how the constants in the estimates depend on the underlying

measure. More precisely, one needs to show that those constants depend only on  $c_1/c_0$  and  $\alpha$  from (26). Again, it is clear that this complication does not arise in the case where  $\mu$  is the two-dimensional Lebesgue measure.

This paper is organized as follows: Auxiliary results on Orlicz spaces and measures are collected in Sec. II. The main results are stated in Sec. III. In Sec. IV, we describe the Birman–Laptev–Solomyak method and then apply it in Sec. V to the Proof of Theorem 3.1. Corollary 3.2 is proved in Sec. VI. The (non)optimality of our main estimate (32) is discussed in Sec. VII. We show that

$$N_-(\mathcal{E}_{\lambda V, \mu, \mathbb{R}^2}) = O(\lambda) \text{ as } \lambda \rightarrow +\infty$$

implies that the first sum in the right-hand side of (32) is finite. Unfortunately, this is not the case for the second sum. However, we show that the Orlicz  $L \log L$  norm, the  $\mathcal{B}$  norm [see (8)] to be more precise, cannot be substituted with a weaker Orlicz norm. Finally, we prove in the Appendix some simple asymptotic results that are needed to justify the applicability of a suitable endpoint trace theorem (Ref. 27, Theorem 11.8; see Theorem 5.1) in our setting (see the Proof of Lemma 5.2).

## II. AUXILIARY MATERIAL

We start by recalling some notions and results from the theory of Orlicz spaces [see, e.g. Refs. 28 (Chap. 8), 29, and 30]. Let  $(\Omega, \Sigma, \mu)$  be a measure space, and let  $\Psi : [0, +\infty) \rightarrow [0, +\infty)$  be a non-decreasing function. The Orlicz class  $K_\Psi(\Omega, \mu)$  is the set of all of measurable functions  $f : \Omega \rightarrow \mathbb{C}$  (or  $\mathbb{R}$ ) such that

$$\int_{\Omega} \Psi(|f(x)|) d\mu(x) < \infty. \tag{7}$$

If  $\Psi(t) = t^p$ ,  $1 \leq p < \infty$ , this is just the  $L^p(\Omega, \mu)$  space.

*Definition 2.1.* A continuous non-decreasing convex function  $\Psi : [0, +\infty) \rightarrow [0, +\infty)$  is called an *N-function* if

$$\lim_{t \rightarrow 0^+} \frac{\Psi(t)}{t} = 0 \text{ and } \lim_{t \rightarrow \infty} \frac{\Psi(t)}{t} = \infty.$$

The function  $\Phi : [0, +\infty) \rightarrow [0, +\infty)$  defined by

$$\Phi(t) := \sup_{s \geq 0} (st - \Psi(s))$$

is called *complementary* to  $\Psi$ .

Examples of complementary functions include

$$\begin{aligned} \Psi(t) = \frac{t^p}{p}, \quad 1 < p < \infty, \quad \Phi(t) = \frac{t^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \\ \mathcal{A}(s) = e^{|s|} - 1 - |s|, \quad \mathcal{B}(s) = (1 + |s|) \ln(1 + |s|) - |s|, \quad s \in \mathbb{R}. \end{aligned} \tag{8}$$

We will use the following notation  $a_+ := \max\{0, a\}$ ,  $a \in \mathbb{R}$ .

*Lemma 2.2* (Ref. 9, Lemma 2.2).  $\frac{1}{2} s \ln_+ s \leq \mathcal{B}(s) \leq s + 2s \ln_+ s$ ,  $\forall s \geq 0$ .

*Definition 2.3.* An *N-function*  $\Psi$  is said to satisfy the *global  $\Delta_2$ -condition* if there exists a positive constant  $k$  such that for every  $t \geq 0$ ,

$$\Psi(2t) \leq k\Psi(t). \tag{9}$$

Similarly,  $\Psi$  is said to satisfy the  *$\Delta_2$ -condition near infinity* if there exists  $t_0 > 0$  such that (9) holds for all  $t \geq t_0$ .

*Definition 2.4.* A pair  $(\Psi, \Omega)$  is called  *$\Delta$ -regular* if either  $\Psi$  satisfies a global  $\Delta_2$ -condition or  $\Psi$  satisfies the  $\Delta_2$ -condition near infinity and  $\mu(\Omega) < \infty$ .

*Lemma 2.5* (Ref. 28, Lemma 8.8).  $K_\Psi(\Omega, \mu)$  is a vector space if and only if  $(\Psi, \Omega)$  is  $\Delta$ -regular.

*Definition 2.6.* The *Orlicz space*  $L_\Psi(\Omega, \mu)$  is the linear span of the Orlicz class  $K_\Psi(\Omega, \mu)$ , that is, the smallest vector space containing  $K_\Psi(\Omega, \mu)$ .

Consequently,  $K_\Psi(\Omega, \mu) = L_\Psi(\Omega, \mu)$  if and only if  $(\Psi, \Omega)$  is  $\Delta$ -regular.

Let  $\Phi$  and  $\Psi$  be mutually complementary  $N$ -functions, and let  $L_\Phi(\Omega, \mu)$  and  $L_\Psi(\Omega, \mu)$  be the corresponding Orlicz spaces. We will use the following norms on  $L_\Psi(\Omega, \mu)$ :

$$\|f\|_{\Psi, \mu} = \|f\|_{\Psi, \Omega, \mu} = \sup \left\{ \left| \int_{\Omega} fg d\mu \right| : \int_{\Omega} \Phi(|g|) d\mu \leq 1 \right\} \quad (10)$$

and

$$\|f\|_{(\Psi, \mu)} = \|f\|_{(\Psi, \Omega, \mu)} = \inf \left\{ \kappa > 0 : \int_{\Omega} \Psi \left( \frac{|f|}{\kappa} \right) d\mu \leq 1 \right\}. \quad (11)$$

These two norms are equivalent,

$$\|f\|_{(\Psi, \mu)} \leq \|f\|_{\Psi, \mu} \leq 2 \|f\|_{(\Psi, \mu)}, \quad \forall f \in L_\Psi(\Omega) \quad (12)$$

[see, e.g., Ref. 29, (9.24)].

Note that

$$\int_{\Omega} \Psi \left( \frac{|f|}{\kappa_0} \right) d\mu \leq C_0, \quad C_0 \geq 1 \Rightarrow \|f\|_{(\Psi, \mu)} \leq C_0 \kappa_0 \quad (13)$$

(see Ref. 9). Indeed, since  $\Psi$  is convex and increasing on  $[0, +\infty)$  and  $\Psi(0) = 0$ , we get for any  $\kappa \geq C_0 \kappa_0$ ,

$$\int_{\Omega} \Psi \left( \frac{|f|}{\kappa} \right) d\mu \leq \int_{\Omega} \Psi \left( \frac{|f|}{C_0 \kappa_0} \right) d\mu \leq \frac{1}{C_0} \int_{\Omega} \Psi \left( \frac{|f|}{\kappa_0} \right) d\mu \leq 1. \quad (14)$$

It follows from (13) with  $\kappa_0 = 1$  that

$$\|f\|_{(\Psi, \mu)} \leq \max \left\{ 1, \int_{\Omega} \Psi(|f|) d\mu \right\}. \quad (15)$$

We will need the following equivalent norm on  $L_\Psi(\Omega, \mu)$  with  $\mu(\Omega) < \infty$ , which was introduced in Ref. 10:

$$\|f\|_{\Psi, \mu}^{(av)} = \|f\|_{\Psi, \Omega, \mu}^{(av)} = \sup \left\{ \left| \int_{\Omega} fg d\mu \right| : \int_{\Omega} \Phi(|g|) d\mu \leq \mu(\Omega) \right\}. \quad (16)$$

*Proposition 2.7* (Ref. 30, Theorem 9.3). For any  $f \in L_\Psi(\Omega, \mu)$  and  $g \in L_\Phi(\Omega, \mu)$ ,

$$\left| \int_{\Omega} fg d\mu \right| \leq \|f\|_{\Psi, \Omega, \mu} \|g\|_{\Phi, \Omega, \mu}. \quad (17)$$

In particular,  $fg \in L^1(\Omega, \mu)$ .

The above is called the Hölder inequality for Orlicz spaces. The following is referred to as the strengthened Hölder inequality:

$$\left| \int_{\Omega} fg d\mu \right| \leq \|f\|_{(\Psi, \Omega, \mu)} \|g\|_{\Phi, \Omega, \mu} \quad (18)$$

for all  $f \in L_\Psi(\Omega, \mu)$  and  $g \in L_\Phi(\Omega, \mu)$  [see Ref. 29, (9.27)].

*Lemma 2.8* (Ref. 10, Lemma 3). For any finite collection of pairwise disjoint subsets  $\Omega_k$  of  $\Omega$ ,

$$\sum_k \|f\|_{\Psi, \Omega_k, \mu}^{(av)} \leq \|f\|_{\Psi, \Omega, \mu}^{(av)}. \quad (19)$$

Let

$$\|f\|_{\Psi, \Omega, \mu}^{(av), \tau} = \sup \left\{ \left| \int_{\Omega} f \varphi d\mu \right| : \int_{\Omega} \Phi(|\varphi|) d\mu \leq \tau \mu(\Omega) \right\}, \quad \tau > 0. \quad (20)$$

Lemma 2.9. For any  $\tau_1, \tau_2 > 0$ ,

$$\min\left\{1, \frac{\tau_2}{\tau_1}\right\} \|f\|_{\Psi, \Omega, \mu}^{(av), \tau_1} \leq \|f\|_{\Psi, \Omega, \mu}^{(av), \tau_2} \leq \max\left\{1, \frac{\tau_2}{\tau_1}\right\} \|f\|_{\Psi, \Omega, \mu}^{(av), \tau_1}. \quad (21)$$

Proof. Let

$$X_1 := \left\{ \varphi : \int_{\Omega} \Phi(|\varphi|) d\mu \leq \tau_1 \mu(\Omega) \right\}, \quad X_2 := \left\{ \varphi : \int_{\Omega} \Phi(|\varphi|) d\mu \leq \tau_2 \mu(\Omega) \right\}.$$

Suppose that  $\tau_1 \leq \tau_2$ . Then, it is clear that  $\|f\|_{\Psi, \Omega, \mu}^{(av), \tau_1} \leq \|f\|_{\Psi, \Omega, \mu}^{(av), \tau_2}$ . Now, since  $\Phi$  is convex and  $\Phi(0) = 0$ ,

$$\varphi \in X_2 \Rightarrow \frac{\tau_1}{\tau_2} \varphi \in X_1 \quad [\text{cf. (14)}].$$

Hence,

$$\|f\|_{\Psi, \Omega, \mu}^{(av), \tau_2} = \sup_{\varphi \in X_2} \left| \int_{\Omega} f \varphi d\mu \right| \leq \sup_{\phi \in X_1} \left| \int_{\Omega} f \cdot \left( \frac{\tau_2}{\tau_1} \phi \right) d\mu \right| = \frac{\tau_2}{\tau_1} \|f\|_{\Psi, \Omega, \mu}^{(av), \tau_1}.$$

On the other hand, suppose that  $\tau_1 \geq \tau_2$ . Then,

$$\|f\|_{\Psi, \Omega, \mu}^{(av), \tau_2} \leq \|f\|_{\Psi, \Omega, \mu}^{(av), \tau_1} \leq \frac{\tau_1}{\tau_2} \|f\|_{\Psi, \Omega, \mu}^{(av), \tau_2}.$$

Hence,

$$\min\left\{1, \frac{\tau_2}{\tau_1}\right\} \|f\|_{\Psi, \Omega, \mu}^{(av), \tau_1} \leq \|f\|_{\Psi, \Omega, \mu}^{(av), \tau_2}$$

and

$$\|f\|_{\Psi, \Omega, \mu}^{(av), \tau_2} \leq \max\left\{1, \frac{\tau_2}{\tau_1}\right\} \|f\|_{\Psi, \Omega, \mu}^{(av), \tau_1}.$$

□

As a result of the above lemma, we have the following:

Corollary 2.10 (Ref. 9, Lemma 2.1).

$$\min\{1, \mu(\Omega)\} \|f\|_{\Psi, \Omega, \mu} \leq \|f\|_{\Psi, \Omega, \mu}^{(av)} \leq \max\{1, \mu(\Omega)\} \|f\|_{\Psi, \Omega, \mu}.$$

Let  $(\Omega_1, \Sigma_1)$  and  $(\Omega_2, \Sigma_2)$  be a pair of measurable spaces and  $\xi : (\Omega_1, \Sigma_1) \rightarrow (\Omega_2, \Sigma_2)$  be an isomorphism, i.e., let  $\xi$  be a bijection such that both  $\xi$  and  $\xi^{-1}$  are measurable. Let  $\mu$  be a finite measure on  $(\Omega_2, \Sigma_2)$  and  $V : (\Omega_2, \Sigma_2) \rightarrow \mathbb{C}$  be a measurable function. Then,  $\tilde{V} := V \circ \xi$  is a measurable function on  $(\Omega_1, \Sigma_1)$  and  $\tilde{\mu} := \mu \circ \xi$ ,

$$\tilde{\mu}(E) = \mu(\xi(E)), \quad E \in \Sigma_1$$

is a measure on  $(\Omega_1, \Sigma_1)$ . For any  $c > 0$  and any mutually complementary  $N$ -functions  $\Phi$  and  $\Psi$ , one gets using (16) and the change of variable formula (see, e.g., Ref. 31, Lemma 5.0.1)

$$\begin{aligned} \|V\|_{\Psi, \Omega_2, \mu}^{(av)} &= \sup\left\{ \left| \int_{\Omega_2} V f d\mu \right| : \int_{\Omega_2} \Phi(|f|) d\mu \leq \mu(\Omega_2) \right\} \\ &= \sup\left\{ \frac{1}{c} \left| \int_{\Omega_1} \tilde{V} g d(c\tilde{\mu}) \right| : \int_{\Omega_1} \Phi(|g|) d(c\tilde{\mu}) \leq c\tilde{\mu}(\Omega_1) \right\} \\ &= \frac{1}{c} \|\tilde{V}\|_{\Psi, \Omega_1, c\tilde{\mu}}^{(av)}. \end{aligned} \quad (22)$$

Hence, by Corollary 2.10,

$$\|\tilde{V}\|_{\Psi, \Omega_1, c\tilde{\mu}} \leq \frac{1}{\min\{1, c\tilde{\mu}(\Omega_1)\}} \|\tilde{V}\|_{\Psi, \Omega_1, c\tilde{\mu}}^{(av)} = \frac{c}{\min\{1, c\tilde{\mu}(\Omega_1)\}} \|V\|_{\Psi, \Omega_2, \mu}^{(av)}. \quad (23)$$

Lemma 2.11.

$$\|f\|_{L_1(\Omega, \mu)} \leq \Psi^{-1}(1) \|f\|_{\Psi, \Omega, \mu}^{(av)}.$$

Proof. Clearly, one only needs to consider the case  $0 < \mu(\Omega) < \infty$ . Let  $\mu_1 := \frac{1}{\mu(\Omega)} \mu$ . Then,  $\mu_1(\Omega) = 1$ , and using (17), Refs. 29 (9.11), and (22) [with  $c = \frac{1}{\mu(\Omega)}$ ,  $(\Omega_1, \Sigma_1) = (\Omega_2, \Sigma_2) = (\Omega, \Sigma)$ , and  $\xi(x) \equiv x$ ], one gets

$$\begin{aligned} \int_{\Omega} |f(x)| d\mu(x) &= \mu(\Omega) \int_{\Omega} |f(x)| d\mu_1(x) \leq \mu(\Omega) \|f\|_{\Psi, \Omega, \mu_1} \|1\|_{\Phi, \Omega, \mu_1} \\ &= \mu(\Omega) \|f\|_{\Psi, \Omega, \mu_1}^{\text{(av)}} \Psi^{-1}(1) = \mu(\Omega) \|f\|_{\Psi, \Omega, \frac{1}{\mu(\Omega)}\mu}^{\text{(av)}} \Psi^{-1}(1) = \|f\|_{\Psi, \Omega, \mu}^{\text{(av)}} \Psi^{-1}(1). \end{aligned}$$

□

Lemma 2.12 (Ref. 9, Lemma 2.5). Let  $\mu(\Omega) > 1$ . Then,

$$\|f\|_{\mathcal{B}, \Omega, \mu}^{\text{(av)}} \leq \|f\|_{\mathcal{B}, \Omega, \mu} + \ln\left(\frac{7}{2} \mu(\Omega)\right) \|f\|_{L_1(\Omega, \mu)}.$$

Lemma 2.13. Let  $\mu$  be a  $\sigma$ -finite Borel measure on  $\mathbb{R}^2$  such that  $\mu(\{x\}) = 0, \forall x \in \mathbb{R}^2$ . Let

$$\Sigma := \{\theta \in [0, \pi) : \exists l_{\theta} \text{ such that } \mu(l_{\theta}) > 0\}, \tag{24}$$

where  $l_{\theta}$  is a line in  $\mathbb{R}^2$  in the direction of the vector  $(\cos \theta, \sin \theta)$ . Then,  $\Sigma$  is at most countable.

Proof. Let

$$\Sigma_N := \{\theta \in [0, \pi) : \exists l_{\theta} \text{ such that } \mu(l_{\theta} \cap B(0, N)) > 0\},$$

where  $B(0, N)$  is the ball of radius  $N \in \mathbb{N}$  centered at 0. Then,

$$\Sigma = \bigcup_{N \in \mathbb{N}} \Sigma_N.$$

It is now enough to show that  $\Sigma_N$  is at most countable for  $\forall N \in \mathbb{N}$ . Suppose that  $\Sigma_N$  is uncountable. Then, there exists a  $\delta > 0$  such that

$$\Sigma_{N, \delta} := \{\theta \in [0, \pi) : \exists l_{\theta} \text{ such that } \mu(l_{\theta} \cap B(0, N)) > \delta\}$$

is infinite. Otherwise,  $\Sigma_N = \bigcup_{n \in \mathbb{N}} \Sigma_{N, \frac{1}{n}}$  would have been finite or countable. Now, take distinct  $\theta_1, \dots, \theta_k, \dots \in \Sigma_{N, \delta}$ . Then,

$$\mu(l_{\theta_k} \cap B(0, N)) > \delta, \quad \forall k \in \mathbb{N}.$$

Since  $l_{\theta_j} \cap l_{\theta_k}, j \neq k$  contains at most one point, then

$$\mu\left(\bigcup_{j \neq k} (l_{\theta_j} \cap l_{\theta_k})\right) = 0.$$

Let

$$\tilde{l}_{\theta_k} := l_{\theta_k} \setminus \bigcup_{j \neq k} (l_{\theta_j} \cap l_{\theta_k}).$$

Then,  $\tilde{l}_{\theta_j} \cap \tilde{l}_{\theta_k} = \emptyset, j \neq k$  and  $\tilde{l}_{\theta_k} \cap B(0, N) \subset B(0, N)$ . So,

$$\sum_{k \in \mathbb{N}} \mu(\tilde{l}_{\theta_k} \cap B(0, N)) = \mu\left(\bigcup_{k \in \mathbb{N}} (\tilde{l}_{\theta_k} \cap B(0, N))\right) \leq \mu(B(0, N)) < \infty.$$

However,

$$\mu(\tilde{l}_{\theta_k} \cap B(0, N)) = \mu(l_{\theta_k} \cap B(0, N)) \geq \delta,$$

which implies

$$\sum_{k \in \mathbb{N}} \mu(\tilde{l}_{\theta_k} \cap B(0, N)) \geq \sum_{k \in \mathbb{N}} \delta = \infty.$$

This contradiction means that  $\Sigma_N$  is at most countable for each  $N \in \mathbb{N}$ . Hence,  $\Sigma$  is at most countable.

□

Corollary 2.14. There exists  $\theta_0 \in [0, \pi/2)$  such that  $\theta_0 \notin \Sigma$  and  $\theta_0 + \frac{\pi}{2} \notin \Sigma$ .

Proof. The set

$$\Sigma - \frac{\pi}{2} := \left\{ \theta - \frac{\pi}{2} : \theta \in \Sigma \right\}$$

is at most countable. This implies that there exists

$$\theta_0 \in [0, \pi/2) \setminus \left( \Sigma \cup \left( \Sigma - \frac{\pi}{2} \right) \right).$$

Thus,  $\theta_0, \theta_0 + \frac{\pi}{2} \notin \Sigma$ . □

Let  $Q$  be an arbitrary unit square with its sides in the directions determined by  $\theta_0$  and  $\theta_0 + \frac{\pi}{2}$  in Corollary 2.14. For a given  $x \in \bar{Q}$  and  $t > 0$ , let  $Q_x(t)$  be the closed square centered at  $x$  with sides of length  $t$  parallel to those of  $Q$ .

**Lemma 2.15** (cf. Lemma 4 in Ref. 10). *Suppose that  $\Psi$  satisfies the  $\Delta_2$ -condition [see (9)]. Then, for every  $f \in L_\Psi(Q, \mu)$ , the function  $t \mapsto \mathcal{J}(t) := \|f\|_{\Psi, Q_x(t), \mu}^{(\text{av})}$  is continuous and  $\mathcal{J}(0+) = 0$ .*

*Proof.* Let  $t > t_0 > 0$ . Take any measurable function  $g$  on  $Q_x(t)$  such that

$$\int_{Q_x(t)} \Phi(|g(x)|) d\mu \leq \mu(Q_x(t))$$

and consider  $h_0 := \rho g$ , where  $\rho = \frac{\mu(Q_x(t_0))}{\mu(Q_x(t))} \leq 1$ . Then,

$$\begin{aligned} \int_{Q_x(t_0)} \Phi(|h_0|) d\mu &= \int_{Q_x(t_0)} \Phi(|\rho g|) d\mu \leq \int_{Q_x(t)} \Phi(|\rho g|) d\mu \\ &= \rho \int_{Q_x(t)} \Phi(|g|) d\mu \leq \rho \mu(Q_x(t)) = \mu(Q_x(t_0)). \end{aligned}$$

Hence,

$$\begin{aligned} 0 &\leq \|f\|_{\Psi, Q_x(t), \mu}^{(\text{av})} - \|f\|_{\Psi, Q_x(t_0), \mu}^{(\text{av})} \\ &= \sup \left\{ \left| \int_{Q_x(t)} fg d\mu \right| : \int_{Q_x(t)} \Phi(|g|) d\mu \leq \mu(Q_x(t)) \right\} \\ &\quad - \sup \left\{ \left| \int_{Q_x(t_0)} fh d\mu \right| : \int_{Q_x(t_0)} \Phi(|h|) d\mu \leq \mu(Q_x(t_0)) \right\} \\ &\leq \sup \left\{ \left| \int_{Q_x(t)} fg d\mu \right| : \int_{Q_x(t)} \Phi(|g|) d\mu \leq \mu(Q_x(t)) \right\} \\ &\quad - \sup \left\{ \rho \left| \int_{Q_x(t_0)} fg d\mu \right| : \int_{Q_x(t)} \Phi(|g|) d\mu \leq \mu(Q_x(t)) \right\} \\ &\leq \sup \left\{ \left| \int_{Q_x(t)} fg d\mu \right| - \rho \left| \int_{Q_x(t_0)} fg d\mu \right| : \int_{Q_x(t)} \Phi(|g(x)|) d\mu \leq \mu(Q_x(t)) \right\} \\ &\leq \sup \left\{ \left| \int_{Q_x(t) \setminus Q_x(t_0)} fg d\mu \right| : \int_{Q_x(t)} \Phi(|g(x)|) d\mu \leq \mu(Q_x(t)) \right\} \\ &\quad + (1 - \rho) \sup \left\{ \left| \int_{Q_x(t_0)} fg d\mu \right| : \int_{Q_x(t)} \Phi(|g(x)|) d\mu \leq \mu(Q_x(t)) \right\}. \end{aligned}$$

For every interval  $I \subseteq Q$  parallel to the sides of  $Q$ ,  $\mu(I) = 0$ . Then,  $\mu(Q_x(t) \setminus Q_x(t_0)) \rightarrow \mu(\partial Q_x(t_0)) = 0$  as  $t \rightarrow t_0$ . Using the Hölder inequality [see (18)], we get

$$\begin{aligned} &\sup \left\{ \left| \int_{Q_x(t) \setminus Q_x(t_0)} fg d\mu \right| : \int_{Q_x(t)} \Phi(|g(x)|) d\mu \leq \mu(Q_x(t)) \right\} \\ &\leq \sup_{\int_{Q_x(t)} \Phi(|g(x)|) d\mu \leq \mu(Q_x(t))} \|f\|_{(\Psi, Q_x(t) \setminus Q_x(t_0), \mu)} \|g\|_{\Phi, Q_x(t) \setminus Q_x(t_0), \mu} \\ &\leq \|f\|_{(\Psi, Q_x(t) \setminus Q_x(t_0), \mu)} 2 \max\{1, \mu(Q_x(t))\} \end{aligned}$$

[see (12) and (15)]. Since  $\Psi$  satisfies the  $\Delta_2$  condition, it follows from Ref. 29, Theorems 9.4 and 10.3 that

$$\lim_{t \rightarrow t_0} \|f\|_{(\Psi, Q_x(t) \setminus Q_x(t_0), \mu)} = 0.$$

Furthermore,

$$\rho = \frac{\mu(Q_x(t_0))}{\mu(Q_x(t))} = 1 - \frac{\mu(Q_x(t) \setminus Q_x(t_0))}{\mu(Q_x(t))} \longrightarrow 1 \text{ as } t \longrightarrow t_0.$$

Hence,

$$(1 - \rho) \sup \left\{ \left| \int_{Q_x(t_0)} f g \, d\mu \right| : \int_{Q_x(t)} \Phi(|g(x)|) \, d\mu \leq \mu(Q_x(t)) \right\} \longrightarrow 0$$

as  $t \longrightarrow t_0$ . The case  $t_0 > t > 0$  is proved similarly.

Finally, the equality  $\mathcal{J}(0+) = 0$  follows from Ref. 29 (Theorems 9.4 and 10.3). □

We will use the following pair of mutually complementary  $N$ -functions:

$$\mathcal{A}(s) = e^{|s|} - 1 - |s|, \quad \mathcal{B}(s) = (1 + |s|) \ln(1 + |s|) - |s|, \quad s \in \mathbb{R}. \tag{25}$$

*Definition 2.16.* Let  $\mu$  be a positive Radon measure on  $\mathbb{R}^2$ . We say the measure  $\mu$  is Ahlfors regular of dimension  $\alpha \in (0, 2]$  if there exist positive constants  $c_0$  and  $c_1$  such that

$$c_0 r^\alpha \leq \mu(B(x, r)) \leq c_1 r^\alpha \tag{26}$$

for all  $0 < r \leq \text{diam}(\text{supp } \mu)$  and all  $x \in \text{supp } \mu$ , where  $B(x, r)$  is a ball of radius  $r$  centered at  $x$  and the constants  $c_0$  and  $c_1$  are independent of the balls.

If the measure  $\mu$  is  $\alpha$ -dimensional Ahlfors regular, then it is equivalent to the  $\alpha$ -dimensional Hausdorff measure (see, e.g., Ref. 32, Lemma 1.2). If  $\text{supp } \mu$  is unbounded, (26) is satisfied for all  $r > 0$ . For more details and examples of unbounded Ahlfors regular sets, see, for example, Refs. 32–34.

Suppose that  $\mu$  is the usual one-dimensional Lebesgue measure on a horizontal or a vertical line. Then, (26) holds with  $\alpha = 1$ . This implies  $\mu(I) \neq 0$  for every nonempty subinterval  $I$  of that line, hence the need of Lemma 2.13 and Corollary 2.14 for the validity of Lemma 2.15 in this case.

Throughout this paper, we consider integrals and Orlicz norms with respect to  $\mu$  over closed rather than open sets. This is because the  $\mu$  measure of the boundary of a set may well be positive.

### III. THE MAIN RESULT

Let  $\mathcal{H}$  be a Hilbert space and let  $\mathbf{q}$  be a Hermitian form with a domain  $\text{Dom}(\mathbf{q}) \subseteq \mathcal{H}$ . Set

$$N_-(\mathbf{q}) := \sup \{ \dim \mathcal{L} \mid \mathbf{q}[u] < 0, \forall u \in \mathcal{L} \setminus \{0\} \}, \tag{27}$$

where  $\mathcal{L}$  denotes a linear subspace of  $\text{Dom}(\mathbf{q})$ . The number  $N_-(\mathbf{q})$  is called the Morse index of  $\mathbf{q}$ . If  $\mathbf{q}$  is the quadratic form of a self-adjoint operator  $A$  with no essential spectrum in  $(-\infty, 0)$ , then by the variational principle,  $N_-(\mathbf{q})$  is the number of negative eigenvalues of  $A$  repeated according to their multiplicity (see, e.g., Ref. 35, S1.3 or Ref. 36, Theorem 10.2.3).

Assume without loss of generality that  $0 \in \text{supp } \mu$  and  $\text{diam}(\text{supp } \mu) > 1$ . Let

$$J_n = [e^{2^{n-1}}, e^{2^n}], \quad n > 0, \quad J_0 := [e^{-1}, e], \quad J_n = [e^{-2^{|n|}}, e^{-2^{|n|-1}}], \quad n < 0$$

and

$$G_n := \int_{|x| \in J_n} |\ln |x|| V(x) \, d\mu(x), \quad n \neq 0, \quad G_0 := \int_{|x| \in J_0} V(x) \, d\mu(x). \tag{28}$$

If  $\text{supp } \mu$  is bounded, there exists  $m \in \mathbb{N}$  such that

$$\left( 2 \frac{c_1}{c_0} \right)^{\frac{m-1}{\alpha}} < \text{diam}(\text{supp } \mu) \leq \left( 2 \frac{c_1}{c_0} \right)^{\frac{m}{\alpha}}.$$

Then, there exists  $\eta$  such that

$$\left( 2 \frac{c_1}{c_0} \right)^{-\frac{1}{\alpha}} < \eta \leq 1 \text{ and } \text{diam}(\text{supp } \mu) = \eta \left( 2 \frac{c_1}{c_0} \right)^{\frac{m}{\alpha}}. \tag{29}$$

If  $\text{supp } \mu$  is unbounded, we just take  $\eta = 1$ . Then, we set

$$Q_n := \left\{ x \in \mathbb{R}^2 : \eta \left( 2 \frac{c_1}{c_0} \right)^{\frac{n-1}{\alpha}} \leq |x| \leq \eta \left( 2 \frac{c_1}{c_0} \right)^{\frac{n}{\alpha}} \right\}, \quad n \in \mathbb{Z} \tag{30}$$

and

$$\mathcal{D}_n := \|V\|_{\mathcal{B}, Q_n, \mu}^{(\text{av})} \tag{31}$$

[see (25)].

Define operator (6) by its quadratic form

$$\begin{aligned} \mathcal{E}_{V\mu, \mathbb{R}^2}[w] &:= \int_{\mathbb{R}^2} |\nabla w(x)|^2 dx - \int_{\mathbb{R}^2} V(x)|w(x)|^2 d\mu(x), \\ \text{Dom}(\mathcal{E}_{V\mu, \mathbb{R}^2}) &= W_2^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2, Vd\mu). \end{aligned}$$

Let  $N_-(\mathcal{E}_{V\mu, \mathbb{R}^2})$  denote the number of negative eigenvalues of (6) counted according to their multiplicities, i.e., the Morse index of  $\mathcal{E}_{V\mu, \mathbb{R}^2}$  defined by (27). Then, we have the following result.

**Theorem 3.1.** *Let  $\mu$  be a positive Radon measure on  $\mathbb{R}^2$  that is Ahlfors regular and  $V \geq 0$ . Then, there exist constants  $A > 0$  and  $c > 0$  such that*

$$N_-(\mathcal{E}_{V\mu, \mathbb{R}^2}) \leq 1 + 4 \sum_{G_n > 1/4} \sqrt{G_n} + A \sum_{\mathcal{D}_n > c} \mathcal{D}_n. \tag{32}$$

*Corollary 3.2.* *Under the conditions of the above theorem, there exists a constant  $B > 0$  such that*

$$N_-(\mathcal{E}_{V\mu, \mathbb{R}^2}) \leq 1 + B \left( \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) d\mu(x) + \|V\|_{\mathcal{B}, \mathbb{R}^2, \mu} \right). \tag{33}$$

The proofs of the theorem and the corollary are given in Secs. V and VI, respectively.

#### IV. THE BIRMAN-LAPTEV-SOLOMYAK METHOD

Our description of the Birman–Solomyak method of estimating  $N_-(\mathcal{E}_V)$  follows.<sup>9,10,25,37</sup>

Let  $(r, \theta)$  denote the polar coordinates in  $\mathbb{R}^2$ ,  $r \in \mathbb{R}_+$ ,  $\theta \in [-\pi, \pi]$ , and

$$w_{\mathcal{R}}(r) := \frac{1}{2\pi} \int_{-\pi}^{\pi} w(r, \theta) d\theta, \quad w_{\mathcal{N}}(r, \theta) := w(r, \theta) - w_{\mathcal{R}}(r), \tag{34}$$

where  $w \in C(\mathbb{R}^2 \setminus \{0\})$ . Then,

$$\int_{-\pi}^{\pi} w_{\mathcal{N}}(r, \theta) d\theta = 0, \quad \forall r > 0, \tag{35}$$

and it is easy to see that

$$\int_{\mathbb{R}^2} w_{\mathcal{R}} v_{\mathcal{N}} dy = 0, \quad \forall w, v \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}).$$

Hence,  $w \mapsto Pw := w_{\mathcal{R}}$  extends to an orthogonal projection  $P : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ .

Using the representation of the gradient in polar coordinates, one gets

$$\begin{aligned} \int_{\mathbb{R}^2} \nabla w_{\mathcal{R}} \nabla v_{\mathcal{N}} dy &= \int_{\mathbb{R}^2} \left( \frac{\partial w_{\mathcal{R}}}{\partial r} \frac{\partial v_{\mathcal{N}}}{\partial r} + \frac{1}{r^2} \frac{\partial w_{\mathcal{R}}}{\partial \theta} \frac{\partial v_{\mathcal{N}}}{\partial \theta} \right) dy \\ &= \int_{\mathbb{R}^2} \frac{\partial w_{\mathcal{R}}}{\partial r} \frac{\partial v_{\mathcal{N}}}{\partial r} dy = \int_{\mathbb{R}^2} \left( \frac{\partial w}{\partial r} \right)_{\mathcal{R}} \left( \frac{\partial v}{\partial r} \right)_{\mathcal{N}} dy = 0, \quad \forall w, v \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}). \end{aligned}$$

Hence,  $P : W_2^1(\mathbb{R}^2) \rightarrow W_2^1(\mathbb{R}^2)$  is also an orthogonal projection.

Since

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla w|^2 dx &= \int_{\mathbb{R}^2} |\nabla w_{\mathcal{R}}|^2 dx + \int_{\mathbb{R}^2} |\nabla w_{\mathcal{N}}|^2 dx, \\ \int_{\mathbb{R}^2} V|w|^2 d\mu(x) &\leq 2 \int_{\mathbb{R}^2} V|w_{\mathcal{R}}|^2 d\mu(x) + 2 \int_{\mathbb{R}^2} V|w_{\mathcal{N}}|^2 d\mu(x), \end{aligned}$$

we have

$$N_-(\mathcal{E}_{V\mu, \mathbb{R}^2}) \leq N_-(\mathcal{E}_{\mathcal{R}, 2V\mu}) + N_-(\mathcal{E}_{\mathcal{N}, 2V\mu}), \tag{36}$$

where  $\mathcal{E}_{\mathcal{R}, 2V\mu}$  and  $\mathcal{E}_{\mathcal{N}, 2V\mu}$  are the restrictions of the form  $\mathcal{E}_{2V\mu, \mathbb{R}^2}$  to  $PW_2^1(\mathbb{R}^2)$  and  $(I - P)W_2^1(\mathbb{R}^2)$ , respectively. Therefore, to estimate  $N_-(\mathcal{E}_{V\mu, \mathbb{R}^2})$ , it is sufficient to find estimates for  $N_-(\mathcal{E}_{\mathcal{R}, 2V\mu})$  and  $N_-(\mathcal{E}_{\mathcal{N}, 2V\mu})$ .

On the space  $PW_2^1(\mathbb{R}^2)$ , a simple exponential change of variables reduces the problem to a one-dimensional Schrödinger operator, which provides an estimate for  $N_-(\mathcal{E}_{\mathcal{R},2V\mu})$  in terms of weighted  $L^1$  norms of  $V$  [see (41) and (42)]. Theorem 7.1 shows that this estimate is optimal in a sense [see also (86)].

On the space  $(I - P)W_2^1(\mathbb{R}^2)$ , one gets an estimate for  $N_-(\mathcal{E}_{\mathcal{N},2V\mu})$  in terms of Orlicz norms of  $V$  [see (78) and (31)]. The variational principle (see, e.g., Ref. 26, Lemma 3.2) implies that

$$N_-(\mathcal{E}_{\mathcal{N},2V\mu}) \leq \sum_{n \in \mathbb{Z}} N_-(\mathcal{E}_{\mathcal{N},2V\mu,Q_n}), \tag{37}$$

where  $Q_n$  are the annuli defined in (30),

$$\begin{aligned} \mathcal{E}_{\mathcal{N},2V\mu,Q_n}[w] &:= \int_{Q_n} |\nabla w(x)|^2 dx - 2 \int_{Q_n} V(x)|w(x)|^2 d\mu(x), \\ \text{Dom}(\mathcal{E}_{\mathcal{N},2V\mu,Q_n}) &= \{w \in (I - P)W_2^1(Q_n) \cap L^2(Q_n, Vd\mu)\}. \end{aligned}$$

The main reason for introducing the space  $(I - P)W_2^1(\mathbb{R}^2)$  is that

$$\int_{Q_n} w(x) dx = 0, \quad \forall w \in (I - P)W_2^1(Q_n) \tag{38}$$

[cf. (35)], which allows one to use the Poincaré inequality and ensures that not all terms in the right-hand side of (37) are necessarily greater or equal to 1.

The Ahlfors condition (26) allows one to obtain estimates for  $N_-(\mathcal{E}_{\mathcal{N},2V\mu,Q_n})$  from those for  $N_-(\mathcal{E}_{\mathcal{N},2V\mu,Q_1})$  by scaling  $x \mapsto x(2\frac{c_1}{c_0})^{\frac{n-1}{\alpha}}$ . So, it is sufficient to find an estimate for  $N_-(\mathcal{E}_{\mathcal{N},2V\mu,Q_1})$ .

### V. PROOF OF THEOREM 3.1

We need to find an estimate for the right-hand side of (36). We start with the first term. Let  $I$  be an arbitrary interval in  $\mathbb{R}_+$ . Define a measure on  $\mathbb{R}_+$  by

$$v(I) := \int_{|x| \in I} V(x) d\mu(x). \tag{39}$$

Then [see (34)],

$$\int_{\mathbb{R}^2} |w_{\mathcal{R}}(x)|^2 V(x) d\mu(x) = \int_{\mathbb{R}_+} |w_{\mathcal{R}}(r)|^2 dv(r).$$

Let  $w \in PW_2^1(\mathbb{R}^2)$ ,  $r = e^t$ ,  $v(t) := w(x) = w_{\mathcal{R}}(r)$  [see (34)]. Then,

$$\int_{\mathbb{R}^2} |\nabla w(x)|^2 dx = 2\pi \int_{\mathbb{R}} |v'(t)|^2 dt$$

and

$$\begin{aligned} \int_{\mathbb{R}^2} V(x)|w(x)|^2 d\mu(x) &= \int_{\mathbb{R}_+} |w_{\mathcal{R}}(r)|^2 dv(r) = \int_{\mathbb{R}} |w_{\mathcal{R}}(e^t)|^2 dv(e^t) \\ &= \int_{\mathbb{R}} |v(t)|^2 dv(e^t). \end{aligned}$$

Let

$$\mathcal{G}_n := \frac{1}{2\pi} \int_{\mathbf{I}_n} |t| dv(e^t), \quad n \neq 0, \quad \mathcal{G}_0 := \frac{1}{2\pi} \int_{\mathbf{I}_0} dv(e^t), \tag{40}$$

where

$$\mathbf{I}_n := [2^{n-1}, 2^n], \quad n > 0, \quad \mathbf{I}_0 := [-1, 1], \quad \mathbf{I}_n := [-2^{|n|}, -2^{|n|-1}], \quad n < 0.$$

Then,

$$N_-(\mathcal{E}_{\mathcal{R},2v}) \leq 1 + 7.61 \sum_{\mathcal{G}_n > 0.046} \sqrt{\mathcal{G}_n}, \tag{41}$$

where

$$\begin{aligned} \mathcal{E}_{\mathcal{R},2v}[v] &:= \int_{\mathbb{R}} |v'(t)|^2 dt - \int_{\mathbb{R}} |v(t)|^2 dv(e^t), \\ \text{Dom}(\mathcal{E}_{\mathcal{R},2v}) &= W_2^1(\mathbb{R}) \cap L^2(\mathbb{R}, dv) \end{aligned}$$

(see Ref. 26). It follows from (28), (39), and (40) that  $G_n = 2\pi\mathcal{G}_n$ , and thus, (41) implies

$$N_-(\mathcal{E}_{\mathcal{R},2\nu\mu}) \leq 1 + 4 \sum_{G_n > 1/4} \sqrt{G_n}. \tag{42}$$

Now, it remains to find an estimate for the second term in the right-hand side of (36) [see (78)]. We begin by stating some auxiliary results.

Let  $\varphi$  be a nonnegative increasing function on  $[0, +\infty)$  such that  $t\varphi(t^{-1})$  decreases and tends to zero as  $t \rightarrow \infty$ . Furthermore, suppose

$$\int_u^{+\infty} t\sigma(t)dt \leq c u\sigma(u) \tag{43}$$

for all  $u > 0$ , where

$$\sigma(v) := v\varphi\left(\frac{1}{v}\right) \tag{44}$$

and  $c$  is a positive constant.

**Theorem 5.1** (Ref. 27, Theorem 11.8). *Let  $\Psi$  and  $\Phi$  be mutually complementary  $N$ -functions and let  $\mu$  be a positive Radon measure on  $\mathbb{R}^2$ . Let  $\varphi$  be the inverse function of  $t \mapsto t\Phi^{-1}(t^{-1})$ , and suppose it satisfies the above conditions. Then, the best, possibly infinite, constant  $A_1$  in*

$$\|w^2\|_{\Psi, \mathbb{R}^2, \mu} \leq A_1 \|w\|_{W_2^1(\mathbb{R}^2)}^2, \quad \forall w \in W_2^1(\mathbb{R}^2) \cap C(\mathbb{R}^2) \tag{45}$$

is equivalent to

$$B_1 = \sup \left\{ \left| \log r \right| \mu(B(x, r)) \Phi^{-1} \left( \frac{1}{\mu(B(x, r))} \right) : x \in \mathbb{R}^2, 0 < r < \frac{1}{2} \right\}, \tag{46}$$

where  $B(x, r)$  is a ball of radius  $r$  centered at  $x$ .

Let  $G \subset \mathbb{R}^2$  be a bounded set with Lipschitz boundary. Then, there exists a bounded linear operator

$$T_G : W_2^1(G) \rightarrow W_2^1(\mathbb{R}^2) \tag{47}$$

such that

$$\begin{aligned} (T_G w)|_G &= w, \quad \forall w \in W_2^1(G), \\ T_G w &\in W_2^1(\mathbb{R}^2) \cap C(\mathbb{R}^2), \quad \forall w \in W_2^1(G) \cap C(\overline{G}) \end{aligned}$$

(see Ref. 38, Chap. VI, Sec. 3).

**Lemma 5.2** (cf. Ref. 27, Corollary 11.8/2). *Consider the complementary  $N$ -functions  $\mathcal{B}(t) = (1+t)\ln(1+t) - t$  and  $\mathcal{A}(t) = e^t - 1 - t$ . Let  $G \subset \mathbb{R}^2$  be a bounded set with Lipschitz boundary. If a positive Radon measure  $\mu$  on  $\overline{G}$  satisfies the following estimate for some  $\alpha > 0$*

$$\mu(B(x, r)) \leq r^\alpha, \quad \forall x \in \overline{G} \text{ and } \forall r \in \left(0, \frac{1}{2}\right), \tag{48}$$

then the inequality

$$\|w^2\|_{\mathcal{A}, \overline{G}, \mu} \leq A_1 \|T_G\|^2 \|w\|_{W_2^1(G)}^2, \quad \forall w \in W_2^1(G) \cap C(\overline{G})$$

holds with a constant  $A_1$  [see (45)] depending only on  $\alpha$ .

*Proof.* First, let us check that the conditions of Theorem 5.1 are satisfied. Let  $\mathfrak{q}(t) := t\mathcal{B}^{-1}\left(\frac{1}{t}\right)$  and  $\frac{1}{t} = \mathcal{B}(s)$ . Then,  $\mathfrak{q}(t) = \frac{s}{\mathcal{B}(s)}$ . Since  $\frac{d}{ds} \left( \frac{\mathcal{B}(s)}{s} \right) = -\frac{1}{s^2} \ln(1+s) + \frac{1}{s} > 0$  for  $s > 0$ , the fraction  $\frac{s}{\mathcal{B}(s)}$  is a decreasing function of  $s$ . It is also clear that  $\frac{s}{\mathcal{B}(s)} \rightarrow 0$  as  $s \rightarrow \infty$ . Hence,  $\mathfrak{q}(t)$  is an increasing function of  $t$  and  $\mathfrak{q}(t) \rightarrow 0$  as  $t \rightarrow 0+$ . Furthermore,

$$\mathfrak{q}(t) = t\mathcal{B}^{-1}\left(\frac{1}{t}\right) = \sqrt{2t}(1 + o(1)) \text{ as } t \rightarrow \infty \tag{49}$$

and

$$\mathfrak{q}(t) = t\mathcal{B}^{-1}\left(\frac{1}{t}\right) = \frac{1}{\ln \frac{1}{t}}(1 + o(1)) \text{ as } t \rightarrow 0 \tag{50}$$

[see (A1) and (A4) in the Appendix].

Let  $\varphi(\tau) := \mathfrak{q}^{-1}(\tau)$ . Then,  $\varphi$  is an increasing function. Let  $x = \mathfrak{q}^{-1}\left(\frac{1}{t}\right)$ . Then,  $x$  is a decreasing function of  $t$ , and  $t = \frac{1}{\mathfrak{q}(x)}$ . Hence,

$$t\varphi(t^{-1}) = t\mathcal{Q}^{-1}\left(\frac{1}{t}\right) = \frac{x}{\mathcal{Q}(x)} = \frac{1}{\mathcal{B}^{-1}\left(\frac{1}{x}\right)}$$

is a decreasing function of  $t$ .

For small values of  $\tau$ ,

$$\varphi(\tau) = \tau e^{-\frac{1}{\tau}} e^{O(1)} \tag{51}$$

[see (A5) and (A8)]. Hence,

$$t\varphi(t^{-1}) = e^{-t} e^{O(1)} \longrightarrow 0 \text{ as } t \longrightarrow \infty$$

and [see (44)]

$$\begin{aligned} \int_u^{+\infty} t\sigma(t) dt &= \int_u^{+\infty} t^2 \varphi\left(\frac{1}{t}\right) dt = \int_u^{+\infty} t e^{-t} e^{O(1)} dt \\ &\leq e^{O(1)} \int_u^{+\infty} t e^{-t} dt = e^{O(1)} (u+1)e^{-u} \leq 2e^{O(1)} u e^{-u} \leq \\ &\leq e^{O(1)} u^2 \varphi\left(\frac{1}{u}\right) = e^{O(1)} u\sigma(u) \text{ as } u \longrightarrow +\infty \end{aligned}$$

[see (51)].

For large values of  $\tau$ ,

$$\varphi(\tau) = \frac{\tau^2}{2} (1 + o(1))$$

[see (49)]. Hence,

$$t\sigma(t) = t^2 \varphi\left(\frac{1}{t}\right) = \frac{1}{2} (1 + o(1)) \text{ as } t \longrightarrow 0+,$$

$$u\sigma(u) \longrightarrow \frac{1}{2} \text{ and } \int_u^{+\infty} t\sigma(t) dt \longrightarrow \text{constant as } u \longrightarrow 0+.$$

Thus,  $\varphi(\tau)$  satisfies condition (43) for all values of  $u$ .

Extend  $\mu$  to  $\mathbb{R}^2$  by  $\mu(E) = 0$  for  $E = \mathbb{R}^2 \setminus \overline{G}$ . It is easy to see that then (48) holds for every  $x \in \mathbb{R}^2$ , and one has the following estimate for the constant  $B_1$  in (46):

$$\begin{aligned} B_1 &= \sup \left\{ \left| \ln r |\mu(B(x, r))| \mathcal{B}^{-1}\left(\frac{1}{\mu(B(x, r))}\right) \right| \mid 0 < r < \frac{1}{2} \right\} \\ &= \sup_{0 < r < \frac{1}{2}} \frac{|\ln r|}{|\ln \mu(B(x, r))|} (1 + o(1)) \leq \text{const} \sup \frac{|\ln r|}{|\ln r^\alpha|} = \frac{\text{const}}{\alpha} \end{aligned}$$

[see (50) and (48)]. Thus, one can take  $A_1 \sim \frac{1}{\alpha}$  in (45). It follows from Theorem 5.1 that

$$\|w\|_{\mathcal{A}, \overline{G}, \mu}^2 = \|(T_G w)\|_{\mathcal{A}, \mathbb{R}^2, \mu}^2 \leq A_1 \|T_G w\|_{W_2^1(\mathbb{R}^2)}^2 \leq A_1 \|T_G\|^2 \|w\|_{W_2^1(G)}^2$$

for all  $w \in W_2^1(G) \cap C(\overline{G})$ . □

We will use the following notation:

$$w_E := \frac{1}{|E|} \int_E w(x) dx, \tag{52}$$

where  $E \subset \mathbb{R}^2$  is a set of a finite Lebesgue measure  $|E|$ .

**Lemma 5.3.** *Let  $G \subset \mathbb{R}^2$  be a bounded set with Lipschitz boundary and  $\mu$  be a positive Radon measure satisfying (48). Then, there exists a constant  $A_2(G) > 0$  such that for any  $V \in L_B(\overline{G}, \mu)$ ,  $V \geq 0$ ,*

$$\int_{\overline{G}} V |w(x)|^2 d\mu(x) \leq A_2(G) \|V\|_{B, \overline{G}, \mu} \|w\|_{W_2^1(G)}^2 \tag{53}$$

for all  $w \in W_2^1(G) \cap C(\overline{G})$  with  $w_G = 0$ . One can take

$$A_2(G) = A_1 \|T_G\|^2 (1 + C_G), \tag{54}$$

where  $A_1$  is the constant from Lemma 5.2 and  $C_G$  is the optimal constant in the Poincaré inequality for  $G$ . In particular, in the case when  $G = Q$  is a unit square with sides chosen in any direction, one can take

$$A_2 = A_2(Q) = A_1 \|T_Q\|^2 (1 + \pi^{-2}), \tag{55}$$

which depends only on  $\alpha$ .

*Proof.* The proof of (53) and (54) follows from the Hölder inequality for Orlicz spaces [see Eq. 17], Lemma 5.2, and the Poincaré inequality (see, e.g., Ref. 39, Chap. IV, Sec. 7, Subsec. 2, Proposition 2). Formula (55) follows from the fact that the best constant in the Poincaré inequality equals  $1/\lambda_2$ , where  $\lambda_2$  is the smallest positive eigenvalue of the Neumann Laplacian (see Ref. 39, Chap. IV, Sec. 7 and Sec. 7, Subsec. 2, Corollary 3)), and that the latter equals  $\pi^2$  for the unit square  $Q$  [see, e.g., Ref. 40, Chap. VIII, Sec. 2, Subsec. 8, (2.398)].  $\square$

*Lemma 5.4.* Suppose  $\mu$  satisfies (26). Let  $\Omega$  be a square centered in the support of  $\mu$  with sides chosen in any direction. Then, there exists a square  $\Omega_0 \subseteq \Omega$  with the same center such that for any  $V \in L_B(\overline{\Omega}, \mu)$ ,  $V \geq 0$ , the following estimate holds:

$$\int_{\overline{\Omega}} V(y) |w(y)|^2 d\mu(y) \leq A_2 \frac{c_1}{c_0} 4^\alpha \|V\|_{B, \overline{\Omega}, \mu}^{(av)} \int_{\Omega} |\nabla w(y)|^2 dy \tag{56}$$

for all  $w \in W_2^1(\Omega) \cap C(\overline{\Omega})$  with  $w_{\Omega_0} = 0$  [see (52)]. Here,  $A_2$  is the same constant as in (55).

*Proof.* Let  $R$  be the side length of  $\Omega$ . It is sufficient to prove (56) in the case  $\frac{R}{2} \leq \text{diam}(\text{supp } \mu)$ . Indeed, if  $\frac{R}{2} > \text{diam}(\text{supp } \mu)$ , then there exists a square  $\Omega_1$  with the same center as  $\Omega$  and with the side length  $R_1$  such that  $R_1 < R$ ,  $\frac{R_1}{2} \leq \text{diam}(\text{supp } \mu)$ , and  $\overline{\Omega_1} \cap \text{supp } \mu = \overline{\Omega} \cap \text{supp } \mu$ . Then, (56) would follow from a similar estimate for  $\Omega_1$  since

$$\int_{\overline{\Omega}} V(y) |w(y)|^2 d\mu(y) = \int_{\overline{\Omega_1}} V(y) |w(y)|^2 d\mu(y) \text{ and } \|V\|_{B, \overline{\Omega}, \mu}^{(av)} = \|V\|_{B, \overline{\Omega_1}, \mu}^{(av)}.$$

Below, we show that in the case  $\frac{R}{2} \leq \text{diam}(\text{supp } \mu)$ , (56) holds with  $\Omega_0 = \Omega$ .

There exist an orthogonal matrix  $U \in \mathbb{R}^{2 \times 2}$  and a vector  $x_0 \in \mathbb{R}^2$  such that  $\Omega = \xi(Q)$ , where  $\xi$  is the similarity transformation  $\xi(y) = RUy + x_0$ ,  $y \in \mathbb{R}^2$ . Let  $\tilde{V} := V \circ \xi$  and  $\tilde{\mu} := \mu \circ \xi$ . Take any  $x \in \overline{Q} \cap \text{supp } \tilde{\mu}$ , i.e., any  $x \in \overline{Q}$  such that  $\xi(x) \in \text{supp } \mu$ . Since  $\xi(B(x, r)) = B(\xi(x), Rr)$  for any  $r > 0$ , (26) implies

$$c_0 (Rr)^\alpha \leq \tilde{\mu}(B(x, r)) = \mu(\xi(B(x, r))) = \mu(B(\xi(x), Rr)) \leq c_1 (Rr)^\alpha \tag{57}$$

for any positive  $r \leq \frac{1}{R} \text{diam}(\text{supp } \mu)$ . It is clear that the latter restriction is not needed for the upper estimate in (57), since  $\mu(B(\xi(x), Rr))$  does not change as  $r$  increases beyond  $\frac{1}{R} \text{diam}(\text{supp } \mu)$ . If  $x \in \overline{Q} \setminus \text{supp } \tilde{\mu}$ , then, obviously,

$$\tilde{\mu}(B(x, r)) = 0, \quad \forall r < \text{dist}(x, \text{supp } \tilde{\mu}).$$

If  $r \geq \text{dist}(x, \text{supp } \tilde{\mu})$ , then there exists  $x_1 \in \text{supp } \tilde{\mu}$  such that  $|x - x_1| \leq r$ . Hence,  $B(x, r) \subset B(x_1, 2r)$ , and it follows from (57) that

$$\tilde{\mu}(B(x, r)) \leq \tilde{\mu}(B(x_1, 2r)) \leq c_1 (2R)^\alpha r^\alpha.$$

Let

$$c := \frac{1}{c_1 (2R)^\alpha}.$$

Then, Lemma 5.3 applies to the measure  $c\tilde{\mu}$ . Using (23) and the equality

$$\int_Q |\nabla(w \circ \xi)(x)|^2 dx = \int_{\Omega} |\nabla w(y)|^2 dy,$$

we get

$$\begin{aligned} \int_{\overline{\Omega}} V(y) |w(y)|^2 d\mu(y) &= \frac{1}{c} \int_{\overline{Q}} V(\xi(x)) |w(\xi(x))|^2 d(c\mu(\xi(x))) \\ &= \frac{1}{c} \int_{\overline{Q}} \tilde{V}(x) |(w \circ \xi)(x)|^2 d(c\tilde{\mu}(x)) \\ &\leq \frac{1}{c} A_2 \|\tilde{V}\|_{B, \overline{Q}, c\tilde{\mu}} \int_Q |\nabla(w \circ \xi)(x)|^2 dx \\ &\leq \frac{1}{c} A_2 \frac{c}{\min\{1, c\tilde{\mu}(\overline{Q})\}} \|V\|_{B, \overline{\Omega}, \mu}^{(av)} \int_{\Omega} |\nabla w(y)|^2 dy. \end{aligned} \tag{58}$$

However,

$$\begin{aligned} \frac{1}{\min\{1, c\bar{\mu}(\bar{Q})\}} &= \max\left\{1, \frac{1}{c\bar{\mu}(\bar{Q})}\right\} = \max\left\{1, \frac{c_1(2R)^\alpha}{\mu(\bar{\Omega})}\right\} \\ &\leq \max\left\{1, \frac{c_1(2R)^\alpha}{c_0\left(\frac{R}{2}\right)^\alpha}\right\} = \frac{c_1}{c_0}4^\alpha. \end{aligned} \tag{59}$$

In the inequality above, we have used (26) and the fact that  $\Omega$  contains a disk of radius  $\frac{R}{2}$  centered in the support of  $\mu$ . Now, (56) follows from (58) and (59).  $\square$

*Remark 5.5.* Estimate (56) may fail if  $\Omega$  is not centered in the support of  $\mu$  (see Ref. 41, Example 3.2.11).

Let  $G \subset \mathbb{R}^2$  be a bounded set with Lipschitz boundary such that  $\mu(\bar{G}) > 0$ . Let  $G_0$  be the smallest closed square containing  $G$  with sides chosen in the directions  $\theta_0$  and  $\theta_0 + \frac{\pi}{2}$  from Corollary 2.14. Since  $\mu(\bar{G}) > 0$ , there exist  $x \in \text{supp } \mu$  such that  $x \in \bar{G} \subseteq G_0$ . Let  $G_1$  be the closed square centered at  $x$  with sides chosen in the same directions as for  $G_0$  and the side length twice that of  $G_0$ . Then,  $G_1 \supset G_0$ . Finally, let  $G^*$  be the closed square with the same center and the same directions of sides as  $G_0$  and with the side length 3 times that of  $G_0$ . Then,

$$\bar{G} \subseteq G_0 \subset G_1 \subset G^*. \tag{60}$$

Since  $G_1$  is centered in  $\text{supp } \mu$ , Lemma 5.4 can be applied to it. On the other hand, an advantage of  $G^*$  is that it does not depend on the choice of  $x \in \text{supp } \mu$  and is uniquely defined by  $G$  once the direction  $\theta_0$  has been chosen. Hence, one can define the following quantity:

$$\kappa_0(G) := \frac{\mu(G^*)}{\mu(\bar{G})}.$$

Furthermore, let

$$V_*(x) := \begin{cases} V(x), & \text{if } x \in \bar{G} \\ 0 & \text{if } x \notin \bar{G}. \end{cases}$$

Then,

$$\|V_*\|_{\mathcal{B}, G_1, \mu}^{(av)} \leq \|V_*\|_{\mathcal{B}, G^*, \mu}^{(av)} = \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av), \kappa_0(G)} \leq \kappa_0(G) \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)} \tag{61}$$

(see Lemma 2.9).

Using the Poincaré inequality (see, e.g., Ref. 39, Chap. IV, Sec. 7, Subsec. 2, Proposition 2), one gets the following estimate for operator (47):

$$\begin{aligned} \|T_G w\|_{W_2^1(G_1)}^2 &\leq \|T_G w\|_{W_2^1(G^*)}^2 \leq \|T_G w\|_{W_2^1(\mathbb{R}^2)}^2 \\ &\leq \|T_G\|^2 \|w\|_{W_2^1(G)}^2 \leq \|T_G\|^2 (1 + C_G) \int_G |\nabla w(x)|^2 dx \end{aligned} \tag{62}$$

for all  $w \in W_2^1(G)$  with  $w_G = 0$ .

*Lemma 5.6.* Let  $\mu$  be a positive Radon measure on  $\mathbb{R}^2$  that is Ahlfors  $\alpha$ -regular, and let  $G \subset \mathbb{R}^2$  be a bounded set with Lipschitz boundary such that  $\mu(\bar{G}) > 0$ . Choose and fix a direction satisfying Corollary 2.14. Furthermore, let  $Q_x(r)$  be the square with sides of length  $r > 0$  in the chosen direction centered at  $x \in \text{supp } \mu \cap \bar{G}$ . Then, for any  $V \in L_{\mathcal{B}}(\bar{G}, \mu)$ ,  $V \geq 0$  and any  $n \in \mathbb{N}$ , there exists a finite cover of  $\text{supp } \mu \cap \bar{G}$  by squares  $Q_{x_k}(r_{x_k})$ ,  $r_{x_k} > 0$ ,  $k = 1, 2, \dots, n_0$  such that  $n_0 \leq n$  and

$$\int_{\bar{G}} V(x) |w(x)|^2 d\mu(x) \leq A_3 n^{-1} \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)} \int_G |\nabla w(x)|^2 dx \tag{63}$$

for all  $w \in W_2^1(G) \cap C(\bar{G})$  with  $(T_G w)_{Q_{x_k}(r_{x_k})} = 0$ ,  $k = 1, \dots, n_0$  and  $w_G = 0$ , where

$$A_3 = C_\alpha \frac{c_1}{c_0} \|T_G\|^2 (1 + C_G) \kappa_0(G)^2 \tag{64}$$

and the constant  $C_\alpha$  depends only on  $\alpha$ .

*Proof.* Let  $N \in \mathbb{N}$  be a bound (see, e.g., Ref. 42, Theorem 2.7) in the Besicovitch covering lemma (see, e.g., Ref. 43, Chap. 1, Theorem 1.1]). If  $n \leq \kappa_0(G)N$ , take  $n_0 = 1$  and let  $Q_{x_1}(r_{x_1})$  be the square  $\Omega_0$  from Lemma 5.4 with  $\Omega = G_1$ . Then, it follows from (56), (61), and (62) that for all  $w \in W_2^1(G) \cap C(\bar{G})$  with  $(T_G w)_{Q_{x_1}(r_{x_1})} = 0$  and  $w_G = 0$ ,

$$\begin{aligned} \int_{\bar{G}} V(x)|w(x)|^2 d\mu(x) &= \int_{G_1} V_*(x)|T_G w(x)|^2 d\mu(x) \\ &\leq A_2 \frac{c_1}{c_0} 4^\alpha \|V_*\|_{\mathcal{B}, G_1, \mu}^{(av)} \int_{G_1} |\nabla(T_G w)(x)|^2 dx \\ &\leq A_2 \frac{c_1}{c_0} 4^\alpha \kappa_0(G) N n^{-1} \|V_*\|_{\mathcal{B}, G^*, \mu}^{(av)} \int_{G^*} |\nabla(T_G w)(x)|^2 dx \\ &\leq A_2 \frac{c_1}{c_0} 4^\alpha \kappa_0(G) N n^{-1} \kappa_0(G) \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)} \|T_G\|^2 (1 + C_G) \int_G |\nabla w(x)|^2 dx \\ &= B_2 n^{-1} \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)} \int_G |\nabla w(x)|^2 dx, \end{aligned} \tag{65}$$

where  $B_2 := A_2 \frac{c_1}{c_0} 4^\alpha \|T_G\|^2 (1 + C_G) \kappa_0(G)^2 N$ .

Now, assume that  $n > \kappa_0(G)N$ . Lemma 2.15 implies that for any  $x \in \text{supp } \mu \cap \bar{G}$ , there is a closed square  $Q_x(r_x)$  centered at  $x$  such that

$$\|V_*\|_{\mathcal{B}, Q_x(r_x), \mu}^{(av)} = \kappa_0(G) N n^{-1} \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)}. \tag{66}$$

Since  $\kappa_0(G)N n^{-1} < 1$ , it is not difficult to see that  $Q_x(r_x) \subseteq G^*$ . Consider the covering  $\Xi = \{Q_x(r_x)\}$  of  $\text{supp } \mu \cap \bar{G}$ . According to the Besicovitch covering lemma,  $\Xi$  has a countable or a finite subcover  $\Xi'$  that can be split into  $N$  subsets  $\Xi'_j, j = 1, \dots, N$  in such a way that the closed squares in each subset are pairwise disjoint. Applying Lemma 2.8 and (61), one gets

$$\begin{aligned} \kappa_0(G) N n^{-1} \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)} \text{card } \Xi'_j &= \sum_{Q_x(r_x) \in \Xi'_j} \|V_*\|_{\mathcal{B}, Q_x(r_x), \mu}^{(av)} \leq \|V_*\|_{\mathcal{B}, G^*, \mu}^{(av)} \\ &\leq \kappa_0(G) \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)}. \end{aligned}$$

Hence,  $\text{card } \Xi'_j \leq n N^{-1}$  and

$$n_0 := \text{card } \Xi' = \sum_{j=1}^N \text{card } \Xi'_j \leq n.$$

Again, using (56), (62), and (66), one gets for all  $w \in W_2^1(G) \cap C(\bar{G})$  with  $(T_G w)_{Q_{x_k}(r_{x_k})} = 0, k = 1, \dots, n_0$  and  $w_G = 0$

$$\begin{aligned} \int_{\bar{G}} V(x)|w(x)|^2 d\mu(x) &= \int_{\text{supp } \mu \cap \bar{G}} V(x)|w(x)|^2 d\mu(x) \\ &\leq \sum_{k=1}^{n_0} \int_{Q_{x_k}(r_{x_k})} V_*(x)|T_G w(x)|^2 d\mu(x) \\ &\leq A_2 \frac{c_1}{c_0} 4^\alpha \sum_{k=1}^{n_0} \|V_*\|_{\mathcal{B}, Q_{x_k}(r_{x_k}), \mu}^{(av)} \int_{Q_{x_k}(r_{x_k})} |\nabla(T_G w)(x)|^2 dx \\ &= A_2 \frac{c_1}{c_0} 4^\alpha \kappa_0(G) N n^{-1} \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)} \sum_{k=1}^{n_0} \int_{Q_{x_k}(r_{x_k})} |\nabla(T_G w)(x)|^2 dx \\ &= A_2 \frac{c_1}{c_0} 4^\alpha \kappa_0(G) N n^{-1} \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)} \sum_{j=1}^N \sum_{Q_{x_k}(r_{x_k}) \in \Xi'_j} \int_{Q_{x_k}(r_{x_k})} |\nabla(T_G w)(x)|^2 dx \\ &\leq A_2 \frac{c_1}{c_0} 4^\alpha \kappa_0(G) N n^{-1} \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)} \sum_{j=1}^N \int_{G^*} |\nabla(T_G w)(x)|^2 dx \\ &\leq A_2 \frac{c_1}{c_0} 4^\alpha \kappa_0(G) N^2 n^{-1} \|T_G\|^2 (1 + C_G) \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)} \int_G |\nabla w(x)|^2 dx \\ &= C_1 n^{-1} \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)} \int_G |\nabla w(x)|^2 dx, \end{aligned}$$

where  $C_1 := A_2 \frac{c_1}{c_0} 4^\alpha \|T_G\|^2 (1 + C_G) \kappa_0(G) N^2$ . It is now left to take

$$A_3 := \max\{B_2, C_1\} = A_2 4^\alpha N \|T_G\|^2 (1 + C_G) \frac{c_1}{c_0} \kappa_0(G) \max\{\kappa_0(G), N\}. \tag{67}$$

□

*Lemma 5.7.* Let  $\mu$  and  $G$  be as in Lemma 5.6. Then,

$$\int_{\overline{G}} V(x) |w(x)|^2 d\mu(x) \leq A_4 \|V\|_{\mathcal{B}, \overline{G}, \mu}^{(av)} \int_G |\nabla w(x)|^2 dx \tag{68}$$

for all  $w \in W_2^1(G) \cap C(\overline{G})$  with  $w_G = 0$ , where

$$A_4 = 2 \|T_G\|^2 (1 + C_G) \left( A_2 \frac{c_1}{c_0} 4^\alpha + \frac{\mathcal{B}^{-1}(1)}{|G|} \right) \kappa_0(G). \tag{69}$$

*Proof.* It follows from (62) that

$$\begin{aligned} |(T_G w)_{G_1}|^2 &= \left| \frac{1}{|G_1|} \int_{G_1} (T_G w)(x) dx \right|^2 \leq \frac{1}{|G_1|} \|T_G w\|_{L_2(G_1)}^2 \\ &\leq \frac{1}{|G|} \|T_G\|^2 (1 + C_G) \int_G |\nabla w(x)|^2 dx. \end{aligned}$$

Using Lemma 2.11, one gets, similar to (65),

$$\begin{aligned} \int_{\overline{G}} V(x) |w(x)|^2 d\mu(x) &= \int_{G_1} V_*(x) |T_G w(x)|^2 d\mu(x) \\ &\leq 2 \int_{G_1} V_*(x) |T_G w(x) - (T_G w)_{G_1}|^2 d\mu(x) \\ &\quad + 2 \int_{G_1} V_*(x) |(T_G w)_{G_1}|^2 d\mu(x) \\ &\leq 2 A_2 \frac{c_1}{c_0} 4^\alpha \|V_*\|_{\mathcal{B}, G_1, \mu}^{(av)} \int_{G_1} |\nabla(T_G w)(x)|^2 dx \\ &\quad + 2 \mathcal{B}^{-1}(1) \|V_*\|_{\mathcal{B}, G_1, \mu}^{(av)} \frac{1}{|G|} \|T_G\|^2 (1 + C_G) \int_G |\nabla w(x)|^2 dx \\ &\leq A_4 \|V\|_{\mathcal{B}, \overline{G}, \mu}^{(av)} \int_G |\nabla w(x)|^2 dx, \end{aligned}$$

where  $A_4$  is given by (69). □

*Remark 5.8.* If  $\mu$  satisfies (26), then the measure  $\frac{1}{c_1} \mu$  satisfies (48). Applying Lemma 5.3 to  $\frac{1}{c_1} \mu$  and using (23) (with  $c = \frac{1}{c_1}$ ,  $\Omega_1 = \Omega_2 = \overline{G}$ , and  $\xi(x) \equiv x$ ), one gets a version of (68) with the following constant:

$$A'_4 = \frac{A_1 \|T_G\|^2 (1 + C_G)}{\min\left\{1, \frac{1}{c_1} \mu(\overline{G})\right\}} \tag{70}$$

in place of  $A_4$ . The terms in (69) and in (64) that depend on the measure  $\mu$  are  $\frac{c_1}{c_0}$  and  $\kappa_0(G)$ . The latter can often be estimated above by a quantity that depends only on  $\frac{c_1}{c_0}$  and  $\alpha$  (see Examples 5.9 and 5.10). On the other hand, (70) contains the term  $\frac{1}{c_1} \mu(\overline{G})$ . Although (70) would also work for us [see (72)], we prefer to use (69) as it matches (64) better than (70).

*Example 5.9.* Let  $\Omega$  be a square centered in the support of  $\mu$  with sides of length  $R$  chosen in any direction. Then, the side length of  $\Omega^*$  does not exceed  $3\sqrt{2}R$ , and

$$\mu(\Omega^*) \leq c_1 (3\sqrt{2}R)^\alpha.$$

If  $\frac{R}{2} \leq \text{diam}(\text{supp } \mu)$ , then

$$\mu(\overline{\Omega}) \geq c_0 \left(\frac{R}{2}\right)^\alpha \text{ and } \kappa_0(\Omega) = \frac{\mu(\Omega^*)}{\mu(\overline{G})} \leq \frac{c_1}{c_0} (6\sqrt{2})^\alpha.$$

If  $\frac{R}{2} > \text{diam}(\text{supp } \mu)$ , then  $\mu(\overline{\Omega}) = \mu(\Omega^*)$  and  $\kappa_0(\Omega) = 1$ .

*Example 5.10.* Let  $G$  be a circular annulus centered at a point  $x$  in the support of  $\mu$  with the radii  $r$  and  $R$  such that

$$\frac{R}{r} \geq \left(2 \frac{c_1}{c_0}\right)^{\frac{1}{\alpha}} \text{ and } R \leq \text{diam}(\text{supp } \mu).$$

Then, the side length of the square  $G^*$  equals  $6R$ , and

$$\begin{aligned} \mu(\overline{G}) &= \mu(\overline{B(x, R)}) - \mu(B(x, r)) \geq c_0 R^\alpha - c_1 r^\alpha \\ &\geq c_0 R^\alpha - c_1 \frac{1}{2} \frac{c_0}{c_1} R^\alpha = \frac{c_0}{2} R^\alpha, \\ \mu(G^*) &\leq c_1 (6R)^\alpha. \end{aligned}$$

Hence,

$$\kappa_0(G) \leq \frac{c_1 (6R)^\alpha}{\frac{c_0}{2} R^\alpha} = 2 \frac{c_1}{c_0} 6^\alpha. \tag{71}$$

Note also that

$$\frac{1}{\min\left\{1, \frac{1}{c_1} \mu(\overline{G})\right\}} \leq \frac{1}{\min\left\{1, \frac{c_0}{2c_1} R^\alpha\right\}} = \max\left\{1, 2 \frac{c_1}{c_0} R^{-\alpha}\right\}. \tag{72}$$

As mentioned above, let  $\mu$  be a positive Radon measure on  $\mathbb{R}^2$  that is Ahlfors  $\alpha$ -regular and let  $G \subset \mathbb{R}^2$  be a bounded set with Lipschitz boundary such that  $\mu(\overline{G}) > 0$ . Let

$$\begin{aligned} \mathcal{E}_{2V\mu, G}[w] &:= \int_G |\nabla w(x)|^2 dx - 2 \int_{\overline{G}} V(x) |w(x)|^2 d\mu(x), \\ \text{Dom}(\mathcal{E}_{2V\mu, G}) &= \{w \in W_2^1(G) \cap L^2(\overline{G}, Vd\mu) \mid w_G = 0\}. \end{aligned} \tag{73}$$

*Lemma 5.11* (cf. Ref. 9, Lemma 7.7).

$$N_-(\mathcal{E}_{2V\mu, G}) \leq A_5 \|V\|_{\mathcal{B}, \overline{G}, \mu}^{(av)} + 2, \quad \forall V \geq 0, \tag{74}$$

where  $A_5 := 2A_3$  and  $A_3$  is the constant in Lemma 5.6.

*Proof.* Let  $n = \left\lceil A_5 \|V\|_{\mathcal{B}, \overline{G}, \mu}^{(av)} \right\rceil + 1$  in Lemma 5.6, where  $\lceil a \rceil$  denotes the largest integer not greater than  $a$ . Take any linear subspace  $\mathcal{L} \subset \text{Dom}(\mathcal{E}_{2V\mu, G})$  such that

$$\dim \mathcal{L} > \left\lceil A_5 \|V\|_{\mathcal{B}, \overline{G}, \mu}^{(av)} \right\rceil + 2.$$

Since  $n_0 \leq n$ , there exists  $w \in \mathcal{L} \setminus \{0\}$  such that  $w_{Q_{x_k}(r_{x_k})} = 0$ ,  $k = 1, \dots, n_0$ , and  $w_G = 0$ . Then,

$$\begin{aligned} \mathcal{E}_{2V\mu, G}[w] &= \int_G |\nabla w(x)|^2 dx - 2 \int_{\overline{G}} V(x) |w(x)|^2 d\mu(x) \\ &\geq \int_G |\nabla w(x)|^2 dx - \frac{A_5 \|V\|_{\mathcal{B}, \overline{G}, \mu}^{(av)}}{\left\lceil A_5 \|V\|_{\mathcal{B}, \overline{G}, \mu}^{(av)} \right\rceil + 1} \int_G |\nabla w(x)|^2 dx \\ &\geq \int_G |\nabla w(x)|^2 dx - \int_G |\nabla w(x)|^2 dx = 0. \end{aligned}$$

Hence,

$$N_-(\mathcal{E}_{2V\mu, G}) \leq \left\lceil A_5 \|V\|_{\mathcal{B}, \overline{G}, \mu}^{(av)} \right\rceil + 2 \leq A_5 \|V\|_{\mathcal{B}, \overline{G}, \mu}^{(av)} + 2. \tag{75}$$

□

*Lemma 5.12.*

$$N_-(\mathcal{E}_{2V\mu, G}) \leq A_6 \|V\|_{\mathcal{B}, \overline{G}, \mu}^{(av)}, \quad \forall V \geq 0, \tag{75}$$

where  $A_6 := 2A_3 + 4A_4$ , and  $A_3$  and  $A_4$  are the constants in (64) and (69), respectively.

*Proof.* By (68),

$$2 \int_{\bar{G}} V(x)|w(x)|^2 d\mu(x) \leq 2A_4 \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)} \int_G |\nabla w(x)|^2 dx$$

for all  $w \in W_2^1(G) \cap C(\bar{G})$  with  $w_G = 0$ .

If  $\|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)} \leq \frac{1}{2A_4}$ , then  $N_-(\mathcal{E}_{2V\mu, G}) = 0$ . If  $\|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)} > \frac{1}{2A_4}$ , then Lemma 5.11 implies

$$N_-(\mathcal{E}_{2V\mu, G}) \leq A_5 \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)} + 2 \leq A_6 \|V\|_{\mathcal{B}, \bar{G}, \mu}^{(av)},$$

where  $A_6 = A_5 + 4A_4 = 2A_3 + 4A_4$ . □

Assume that  $0 \in \text{supp } \mu$ . Let  $\mathbb{Z}_\mu := \mathbb{Z}$  if  $\text{supp } \mu$  is unbounded and  $\mathbb{Z}_\mu := \mathbb{Z} \cap (-\infty, m]$  if  $\text{supp } \mu$  is bounded [see (29)].

*Lemma 5.13.* *There exists a constant  $A_8 > 0$  such that*

$$N_-(\mathcal{E}_{\mathcal{N}, 2V\mu, Q_n}) \leq A_8 \|V\|_{\mathcal{B}, Q_n, \mu}^{(av)}, \quad \forall V \geq 0, \quad \forall n \in \mathbb{Z}_\mu \tag{76}$$

[see (73) and (30)].

*Proof.* We start with the case  $n = 1$ . It follows from Lemma 5.12 and Example 5.10 that

$$N_-(\mathcal{E}_{\mathcal{N}, 2V\mu, Q_1}) \leq A_8 \|V\|_{\mathcal{B}, Q_1, \mu}^{(av)}, \quad \forall V \geq 0, \tag{77}$$

with

$$\begin{aligned} A_8 &= 2C_\alpha \frac{c_1}{c_0} \|T_{Q_1}\|^2 (1 + C_{Q_1}) \left(2 \frac{c_1}{c_0} 6^\alpha\right)^2 \\ &\quad + 8 \|T_{Q_1}\|^2 (1 + C_{Q_1}) \left(A_2 \frac{c_1}{c_0} 4^\alpha + \frac{\mathcal{B}^{-1}(1)}{|G_1|}\right) 2 \frac{c_1}{c_0} 6^\alpha \\ &= 8C_\alpha 6^{2\alpha} \left(\frac{c_1}{c_0}\right)^3 \|T_{Q_1}\|^2 (1 + C_{Q_1}) \\ &\quad + 16 \|T_{Q_1}\|^2 (1 + C_{Q_1}) \left(A_2 \frac{c_1}{c_0} 4^\alpha + \frac{\mathcal{B}^{-1}(1)}{|G_1|}\right) \frac{c_1}{c_0} 6^\alpha. \end{aligned}$$

As far as the dependence on the measure  $\mu$  is concerned,  $A_8$  depends only on the ratio  $\frac{c_1}{c_0}$ .

Let  $\xi : Q_1 \rightarrow Q_n$  be given by  $\xi(x) := x \left(2 \frac{c_1}{c_0}\right)^{\frac{n-1}{\alpha}}$ . Let  $\tilde{V} := V \circ \xi$ ,  $\tilde{\mu} := \mu \circ \xi$  and  $\tilde{w} := w \circ \xi$ . Since  $\xi(B(x, r)) = B\left(\xi(x), \left(2 \frac{c_1}{c_0}\right)^{\frac{n-1}{\alpha}} r\right)$  for any  $r > 0$ ,  $\tilde{\mu}$  satisfies the following analog of (26) [cf. (57)]:

$$\tilde{c}_0 r^\alpha \leq \tilde{\mu}(B(x, r)) \leq \tilde{c}_1 r^\alpha$$

for all  $0 < r \leq \text{diam}(\text{supp } \tilde{\mu})$ , where  $\tilde{c}_0 := c_0 \left(2 \frac{c_1}{c_0}\right)^{n-1}$ ,  $\tilde{c}_1 := c_1 \left(2 \frac{c_1}{c_0}\right)^{n-1}$ , and  $\frac{\tilde{c}_1}{\tilde{c}_0} = \frac{c_1}{c_0}$ . Now,

$$\begin{aligned} &\int_{Q_n} |\nabla w(y)|^2 dy - 2 \int_{Q_n} V(y)|w(y)|^2 d\mu(y) \\ &= \int_{Q_1} |\nabla \tilde{w}(x)|^2 dx - 2 \int_{Q_1} \tilde{V}(x)|\tilde{w}(x)|^2 d\tilde{\mu}(x). \end{aligned}$$

It follows from (77) that

$$N_-(\mathcal{E}_{\mathcal{N}, 2V\mu, Q_n}) = N_-(\mathcal{E}_{\mathcal{N}, 2\tilde{V}\tilde{\mu}, Q_1}) \leq A_8 \|\tilde{V}\|_{\mathcal{B}, Q_1, \tilde{\mu}}^{(av)}, \quad \forall \tilde{V} \geq 0.$$

It follows from (22) with  $c = 1$  that  $\|\tilde{V}\|_{\mathcal{B}, Q_1, \tilde{\mu}}^{(av)} = \|V\|_{\mathcal{B}, Q_n, \mu}^{(av)}$ . Thus,

$$N_-(\mathcal{E}_{\mathcal{N}, 2V\mu, Q_n}) \leq A_8 \|V\|_{\mathcal{B}, Q_n, \mu}^{(av)}, \quad \forall V \geq 0.$$

Hence, the scaling  $x \mapsto x \left(2 \frac{c_1}{c_0}\right)^{\frac{n-1}{\alpha}}$  allows one to reduce the case of any  $n \in \mathbb{Z}_\mu$  to the case  $n = 1$ . □

We are now in position to derive an estimate for the second term in the right-hand side of (36) from the variational principle (see, e.g., Ref. 26, Lemma 3.2). Note that  $\text{supp } \mu \setminus \{0\} \subseteq \cup_{n \in \mathbb{Z}_\mu} Q_n$  and  $\mu(\{0\}) = 0$ , and that (38) implies

$$w|_{Q_n} \in \text{Dom}(\mathcal{E}_{V\mu, Q_n}), \quad \forall w \in \text{Dom}(\mathcal{E}_{\mathcal{N}, 2V\mu}).$$

Hence, the above lemma implies, for any  $c < \frac{1}{A_8}$ ,

$$N_-(\mathcal{E}_{\mathcal{N}, 2V\mu}) \leq A_8 \sum_{D_n > c} \mathcal{D}_n, \quad \forall V \geq 0 \tag{78}$$

[see (31)]. Thus, Theorem 3.1 follows from (36), (42), and (78).

### VI. PROOF OF COROLLARY 3.2

It is easy to see that

$$\sum_{G_n > 1/4} \sqrt{G_n} \leq \sum_{G_n > 1/4} 2G_n \leq 2 \sum_{n \in \mathbb{Z}} G_n. \tag{79}$$

Let  $\Omega_{-1}$  be the closed disc  $\overline{B(0, e^{-1})}$  and  $\beta \in (0, \alpha)$ . Then, using (18), (26), and Fubini's theorem, one gets

$$\begin{aligned} \sum_{n < 0} G_n &\leq 2 \int_{|x| \leq 1/e} V(x) |\ln |x|| \, d\mu(x) \leq 2 \|V\|_{\mathcal{B}, \Omega_{-1}, \mu} \|\ln |\cdot|\|_{(\mathcal{A}, \Omega_{-1}, \mu)}, \\ &\int_{\Omega_{-1}} \mathcal{A}(\beta |\ln |x||) \, d\mu(x) \leq \int_{|x| \leq 1/e} e^{\frac{\ln |x|}{|\beta|}} \, d\mu(x) \leq \int_{|x| \leq 1} \frac{1}{|x|^\beta} \, d\mu(x) \\ &= \int_{|x| \leq 1} \left( \beta \int_{|x|}^1 r^{-\beta-1} \, dr + 1 \right) \, d\mu(x) \\ &= \beta \int_0^1 r^{-\beta-1} \int_{|x| \leq r} \, d\mu(x) \, dr + \int_{|x| \leq 1} 1 \, d\mu(x) \\ &= \beta \int_0^1 r^{-\beta-1} \mu(B(0, r)) \, dr + \mu(B(0, 1)) \leq \beta \int_0^1 r^{-\beta-1} c_1 r^\alpha \, dr + c_1 \\ &= c_1 \left( \frac{\beta}{\alpha - \beta} + 1 \right) = c_1 \frac{\alpha}{\alpha - \beta} =: A_9. \end{aligned}$$

{We have  $\sum_{n < 0} G_n \leq 2 \int_{|x| \leq 1/e} \dots$  rather than  $\sum_{n < 0} G_n = \int_{|x| \leq 1/e} \dots$  in the first inequality above because  $G_n$  are integrals over domains with intersections that may have positive measure  $\mu$  [see (28)],

$$\mu(\{x \in \mathbb{R}^2 \mid |x| \in J_{n-1}\} \cap \{x \in \mathbb{R}^2 \mid |x| \in J_n\}) = \mu(\{x \in \mathbb{R}^2 \mid |x| = e^{-2|n|}\}),$$

which may be positive. A similar situation occurs in (84) and in the Proof of Lemma 6.1.} Hence,

$$\|\ln |\cdot|\|_{(\mathcal{A}, \Omega_{-1}, \mu)} \leq \frac{1}{\beta} \max\{1, A_9\} =: A_{10}$$

[see (13)] and

$$\sum_{n < 0} G_n \leq 2A_{10} \|V\|_{\mathcal{B}, \Omega_{-1}, \mu} \leq 2A_{10} \|V\|_{\mathcal{B}, \mathbb{R}^2, \mu}. \tag{80}$$

Furthermore,

$$\begin{aligned} G_0 &= \int_{e^{-1} \leq |x| \leq e} V(x) \, d\mu(x) \\ &\leq \frac{1}{\ln(1 + e^{-1})} \int_{e^{-1} \leq |x| \leq e} V(x) \ln(1 + |x|) \, d\mu(x) \\ &\leq \frac{1}{\ln(1 + e^{-1})} \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) \, d\mu(x) \end{aligned} \tag{81}$$

and

$$\sum_{n > 0} G_n \leq 2 \int_{|x| \geq e} V(x) \ln |x| \, d\mu(x) \leq 2 \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) \, d\mu(x). \tag{82}$$

It follows from (79)–(82) that

$$\sum_{G_n > 1/4} \sqrt{G_n} \leq A_{11} \left( \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) \, d\mu(x) + \|V\|_{\mathcal{B}, \mathbb{R}^2, \mu} \right), \tag{83}$$

where

$$A_{11} = 2 \max \left\{ 2A_{10}, \frac{1}{\ln(1 + e^{-1})} + 2 \right\}.$$

Let  $\Omega_0$  be the closed unit disc  $\overline{B(0, 1)}$ . It follows from Lemma 2.8 and Corollary 2.10 that

$$\begin{aligned} \sum_{n \leq 0} \mathcal{D}_n &= \sum_{k \leq 0} \mathcal{D}_{2k} + \sum_{k \leq 0} \mathcal{D}_{2k-1} \leq 2 \|V\|_{\mathcal{B}, \Omega_0, \mu}^{(av)} \\ &\leq 2 \max\{1, \mu(\Omega_0)\} \|V\|_{\mathcal{B}, \Omega_0, \mu} \leq 2 \max\{1, \mu(\Omega_0)\} \|V\|_{\mathcal{B}, \mathbb{R}^2, \mu}. \end{aligned} \tag{84}$$

We need the following lemma to estimate  $\sum_{n \geq 1} \mathcal{D}_n$ .

*Lemma 6.1* (cf. Ref. 9, Lemma 8.1). *There exists  $A_{12} > 0$  such that*

$$\sum_{n=1}^{\infty} \|V\|_{\mathcal{B}, Q_n, \mu} \leq A_{12} \left( \|V\|_{\mathcal{B}, \mathbb{R}^2 \setminus B(0,1), \mu} + \int_{|x| \geq 1} V(x) \ln(2 + \ln|x|) d\mu(x) \right)$$

for any  $V \geq 0$ .

*Proof.* Suppose first that  $\|V\|_{(\mathcal{B}, \mathbb{R}^2 \setminus B(0,1), \mu)} = 1$  and let

$$\alpha_n := \int_{Q_n} \mathcal{B}(V(x)) d\mu(x), \quad \kappa_n := \|V\|_{(\mathcal{B}, Q_n, \mu)}, \quad n \in \mathbb{N}.$$

Then,

$$\begin{aligned} \kappa_n &\leq \|V\|_{(\mathcal{B}, \mathbb{R}^2 \setminus B(0,1), \mu)} = 1, \\ \sum_{n=1}^{\infty} \alpha_n &= \sum_{n=1}^{\infty} \int_{Q_n} \mathcal{B}(V(x)) d\mu(x) \leq 2 \int_{\mathbb{R}^2 \setminus B(0,1)} \mathcal{B}(V(x)) d\mu(x) = 2, \end{aligned}$$

and it follows from Lemma 2.2 that

$$\begin{aligned} 1 &= \int_{Q_n} \mathcal{B}\left(\frac{V(x)}{\kappa_n}\right) d\mu(x) \leq \int_{Q_n} \left(\frac{V(x)}{\kappa_n} + 2 \frac{V(x)}{\kappa_n} \ln_+ \frac{V(x)}{\kappa_n}\right) d\mu(x) \\ &\leq \frac{1}{\kappa_n} \int_{Q_n} (V(x) + 2V(x) \ln_+ V(x)) d\mu(x) + \frac{2}{\kappa_n} \ln \frac{1}{\kappa_n} \|V\|_{L_1(Q_n, \mu)} \\ &\leq \frac{4}{\kappa_n} \alpha_n + \frac{1}{\kappa_n} \left(1 + 2 \ln \frac{1}{\kappa_n}\right) \|V\|_{L_1(Q_n, \mu)}. \end{aligned}$$

Hence,

$$\kappa_n \leq 4\alpha_n + \left(1 + 2 \ln \frac{1}{\kappa_n}\right) \|V\|_{L_1(Q_n, \mu)}$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \|V\|_{\mathcal{B}, Q_n, \mu} &\leq 2 \sum_{n=1}^{\infty} \kappa_n = 2 \sum_{\kappa_n \leq 1/n^2} \kappa_n + 2 \sum_{\kappa_n > 1/n^2} \kappa_n \\ &\leq 2 \sum_{n=1}^{\infty} \frac{1}{n^2} + 8 \sum_{n=1}^{\infty} \alpha_n + 2 \sum_{n=1}^{\infty} (1 + 4 \ln n) \|V\|_{L_1(Q_n, \mu)} \\ &\leq \frac{\pi^2}{3} + 16 + 2 \sum_{n=1}^{\infty} (1 + 4 \ln n) \int_{\left(2 \frac{\ln 1}{e_0}\right)^{\frac{n-1}{\alpha}} \leq |x| \leq \left(2 \frac{\ln 1}{e_0}\right)^{\frac{n}{\alpha}}} V(x) d\mu(x) \\ &\leq \frac{\pi^2}{3} + 16 + A_{13} \sum_{n=1}^{\infty} \int_{\left(2 \frac{\ln 1}{e_0}\right)^{\frac{n-1}{\alpha}} \leq |x| \leq \left(2 \frac{\ln 1}{e_0}\right)^{\frac{n}{\alpha}}} V(x) \ln(2 + \ln|x|) d\mu(x) \\ &\leq \frac{\pi^2}{3} + 16 + 2A_{13} \int_{|x| \geq 1} V(x) \ln(2 + \ln|x|) d\mu(x) \\ &\leq A_{12} \left( \|V\|_{\mathcal{B}, \mathbb{R}^2 \setminus B(0,1), \mu} + \int_{|x| \geq 1} V(x) \ln(2 + \ln|x|) d\mu(x) \right) \end{aligned}$$

[see (12)]. The case of a general  $V$  is reduced to  $\|V\|_{(\mathcal{B}, \mathbb{R}^2 \setminus B(0,1), \mu)} = 1$  by the scaling  $V \mapsto tV$ ,  $t > 0$ . □

Using Lemmas 2.12 and 6.1 (see also Corollary 2.10), one gets

$$\begin{aligned} \sum_{n \geq 1} \mathcal{D}_n &= \sum_{n \geq 1} \|V\|_{\mathcal{B}, Q_n, \mu}^{(av)} \\ &\leq \sum_{n=1}^{\infty} \|V\|_{\mathcal{B}, Q_n, \mu} + \sum_{n=1}^{\infty} \max\left\{0, \ln\left(\frac{7}{2} \mu(Q_n)\right)\right\} \int_{Q_n} V(x) d\mu(x) \\ &\leq \sum_{n=1}^{\infty} \|V\|_{\mathcal{B}, Q_n, \mu} + \sum_{n=1}^{\infty} \max\left\{0, \ln\left(\frac{7}{2} \left(2 \frac{c_1}{c_0}\right)^n\right)\right\} \int_{Q_n} V(x) d\mu(x) \\ &\leq \sum_{n=1}^{\infty} \|V\|_{\mathcal{B}, Q_n, \mu} + A_{14} \sum_{n=1}^{\infty} n \int_{\left(2 \frac{c_1}{c_0}\right)^{\frac{n-1}{\alpha}} \leq |x| \leq \left(2 \frac{c_1}{c_0}\right)^{\frac{n}{\alpha}}} V(x) d\mu(x) \\ &\leq \sum_{n=1}^{\infty} \|V\|_{\mathcal{B}, Q_n, \mu} + A_{15} \sum_{n=1}^{\infty} \int_{\left(2 \frac{c_1}{c_0}\right)^{\frac{n-1}{\alpha}} \leq |x| \leq \left(2 \frac{c_1}{c_0}\right)^{\frac{n}{\alpha}}} V(x) \ln(1 + |x|) d\mu(x) \\ &\leq A_{16} \left( \|V\|_{\mathcal{B}, \mathbb{R}^2 \setminus B(0,1), \mu} + \int_{|x| \geq 1} V(x) \ln(2 + \ln |x|) d\mu(x) \right. \\ &\quad \left. + \int_{|x| \geq 1} V(x) \ln(1 + |x|) d\mu(x) \right) \\ &\leq A_{17} \left( \|V\|_{\mathcal{B}, \mathbb{R}^2, \mu} + \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) d\mu(x) \right), \quad \forall V \geq 0. \end{aligned}$$

Hence, it follows from (84) that

$$\sum_{n \in \mathbb{Z}} \mathcal{D}_n \leq A_{18} \left( \|V\|_{\mathcal{B}, \mathbb{R}^2, \mu} + \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) d\mu(x) \right). \tag{85}$$

Estimate (33) now follows from Theorem 3.1 and (83), (85).

### VII. CONCLUDING REMARKS

For a sequence of numbers  $(a_n)_{n \in \mathbb{Z}}$ , let

$$\|(a_n)_{n \in \mathbb{Z}}\|_{1, \infty} := \sup_{s > 0} (s \operatorname{card}\{n : |a_n| > s\}).$$

It is easy to see that

$$\|(a_n)_{n \in \mathbb{Z}}\|_{1, \infty} \leq \|(a_n)_{n \in \mathbb{Z}}\|_1 = \sum_{n \in \mathbb{Z}} |a_n|.$$

In addition,

$$\sum_{|a_n| > c} \sqrt{|a_n|} \leq \frac{2}{\sqrt{c}} \|(a_n)_{n \in \mathbb{Z}}\|_{1, \infty} \tag{86}$$

and

$$\sum_{|a_n| > c} \sqrt{\gamma |a_n|} = O(\gamma) \text{ as } \gamma \rightarrow +\infty \iff \|(a_n)_{n \in \mathbb{Z}}\|_{1, \infty} < \infty \tag{87}$$

[see Ref. 9, (49), (77), and (78)].

**Theorem 7.1.** *Let  $V \geq 0$ . If  $N_-(\mathcal{E}_{\gamma V, \mu, \mathbb{R}^2}) = O(\gamma)$  as  $\gamma \rightarrow +\infty$ , then  $\|(G_n)_{n \in \mathbb{Z}}\|_{1, \infty} < \infty$ .*

*Proof.* This follows by replacing the Lebesgue measure with  $\mu$  in the proofs of Ref. 9, Theorems 9.1 and 9.2. □

The above theorem and (86) show that the term  $\sum_{G_n > 1/4} \sqrt{G_n}$  in (32) is optimal in a sense. Although the same cannot be said about the term  $\sum_{\mathcal{D}_n > c} \mathcal{D}_n$ , the following theorem shows that it is optimal in the class of Orlicz norms. More precisely, no estimate of the type

$$N_-(\mathcal{E}_{V, \mu, \mathbb{R}^2}) \leq \operatorname{const} + \int_{\mathbb{R}^2} V(x) W(x) d\mu(x) + \operatorname{const} \|V\|_{\Psi, \mathbb{R}^2, \mu} \tag{88}$$

can hold with a norm  $\|V\|_{\Psi, \mathbb{R}^2, \mu}$  weaker than  $\|V\|_{\mathcal{B}, \mathbb{R}^2, \mu}$ , provided that the weight function  $W$  is bounded in a neighborhood of at least one point in the support of  $\mu$ .

**Theorem 7.2** (cf. Ref. 9, Theorem 9.4). *Let  $W \geq 0$  be bounded in a neighborhood of at least one point in the support of  $\mu$ , and let  $\Psi$  be an  $N$ -function such that*

$$\lim_{s \rightarrow \infty} \frac{\Psi(s)}{\mathcal{B}(s)} = 0.$$

Then, there exists a compactly supported  $V \geq 0$  such that

$$\int_{\mathbb{R}^2} V(x)W(x) d\mu(x) + \|V\|_{\Psi, \mathbb{R}^2, \mu} < \infty$$

and  $N_-(\mathcal{E}_{V, \mu, \mathbb{R}^2}) = \infty$ .

*Proof.* Shifting the independent variable if necessary, we can assume that  $0 \in \text{supp } \mu$  and  $W$  is bounded in a neighborhood of 0. Let  $r_0 > 0$  be such that  $W$  is bounded in the open ball  $B(0, r_0)$ .

Let

$$\beta(s) := \sup_{t \geq s} \frac{\Psi(t)}{\mathcal{B}(t)}.$$

Then,  $\beta$  is a non-increasing function,  $\beta(s) \rightarrow 0$  as  $s \rightarrow \infty$ , and  $\Psi(s) \leq \beta(s)\mathcal{B}(s)$ . Since  $\Psi$  is an  $N$ -function,  $\Psi(s)/s \rightarrow \infty$  as  $s \rightarrow \infty$  (see Sec. II). Hence, there exists  $s_0 \geq e^{\frac{1}{\alpha}} > 1$  such that  $\Psi(s) \geq s$  and  $\beta(s) \leq 1$  for  $s \geq s_0^\alpha$ . Choose  $\rho_k \in (0, 1/s_0)$  in such a way that

$$\sum_{k=1}^{\infty} \beta\left(\frac{1}{\rho_k^\alpha}\right) < \infty.$$

It follows from (26) that  $\forall r > 0$ , the disk  $B(0, r)$  contains points of the support of  $\mu$  different from 0. Let  $x^{(1)} \in \text{supp } \mu \setminus \{0\}$  be such that

$$|x^{(1)}| < \min\left\{\frac{2}{3}r_0, 2\rho_1\right\}.$$

One can choose  $x^{(k)}$ ,  $k \in \mathbb{N}$  inductively as follows: suppose  $x^{(1)}, \dots, x^{(k)} \in \text{supp } \mu \setminus \{0\}$  have been chosen. Take  $x^{(k+1)} \in \text{supp } \mu \setminus \{0\}$  such that

$$|x^{(k+1)}| < \min\left\{\frac{1}{3}|x^{(k)}|, 2\rho_{k+1}\right\}.$$

Since  $|x^{(k+1)}| < \frac{1}{3}|x^{(k)}|$ , it is easy to see that the open disks  $B(x^{(k)}, \frac{1}{2}|x^{(k)}|)$ ,  $k \in \mathbb{N}$ , lie in  $B(0, r_0)$  and are pairwise disjoint. Let  $r_k := \frac{1}{2}|x^{(k)}|$ . Then,  $r_k < \rho_k$ ,  $k \in \mathbb{N}$ . For a constant  $A_{19} > 0$  to be specified later, let

$$V(x) := \begin{cases} t_k, & x \in B(x^{(k)}, r_k^2), k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Since the function  $r \mapsto r^\alpha \ln \frac{1}{r}$  has maximum equal to  $\frac{1}{\alpha e}$ , one can choose  $A_{19} > 0$  such that  $A_{19}\alpha e > 1$  and

$$t_k = \frac{A_{19}}{\ln \frac{1}{r_k}} r_k^{-2\alpha} = \frac{A_{19}}{r_k^\alpha \ln \frac{1}{r_k}} r_k^{-\alpha} > \frac{1}{r_k^\alpha} > \frac{1}{\rho_k^\alpha} > s_0^\alpha \geq e.$$

Then,

$$\begin{aligned} \int_{\mathbb{R}^2} \Psi(V(x)) d\mu(x) &= \sum_{k=1}^{\infty} \Psi(t_k) \mu(B(x^{(k)}, r_k^2)) \leq \sum_{k=1}^{\infty} \Psi(t_k) c_1 r_k^{2\alpha} \\ &\leq c_1 \sum_{k=1}^{\infty} r_k^{2\alpha} \beta(t_k) \mathcal{B}(t_k) \leq c_1 \sum_{k=1}^{\infty} r_k^{2\alpha} \beta(t_k) (1+t_k) \ln(1+t_k) \\ &< 4c_1 \sum_{k=1}^{\infty} r_k^{2\alpha} \beta(t_k) t_k \ln t_k = 4c_1 \sum_{k=1}^{\infty} \beta(t_k) \frac{A_{19}}{\ln \frac{1}{r_k}} \ln \frac{A_{19}}{r_k^{2\alpha} \ln \frac{1}{r_k}} \\ &\leq 4c_1 A_{19} \sum_{k=1}^{\infty} \beta\left(\frac{1}{r_k^\alpha}\right) \frac{1}{\ln \frac{1}{r_k}} \ln \frac{A_{19}\alpha}{r_k^{2\alpha}} \\ &\leq \text{const} \sum_{k=1}^{\infty} \beta\left(\frac{1}{r_k^\alpha}\right) \leq \text{const} \sum_{k=1}^{\infty} \beta\left(\frac{1}{\rho_k^\alpha}\right) < \infty. \end{aligned}$$

Thus,  $\|V\|_{\Psi, \mathbb{R}^2, \mu} < \infty$  [see (15) and (12)]. Since  $t_k > \frac{1}{r_k} > s_0^\alpha$ , one has  $t_k \leq \Psi(t_k)$  and

$$\int_{\mathbb{R}^2} V(x) d\mu(x) \leq \int_{\mathbb{R}^2} \Psi(V(x)) d\mu(x) < \infty.$$

Since  $W$  is bounded in  $B(0, r_0)$ ,

$$\int_{\mathbb{R}^2} V(x)W(x) d\mu(x) < \infty.$$

Let

$$w_k(x) := \begin{cases} 1, & |x - x^{(k)}| \leq r_k^2 \\ \frac{\ln(r_k/|x - x^{(k)}|)}{\ln(1/r_k)}, & r_k^2 < |x - x^{(k)}| \leq r_k \\ 0, & |x - x^{(k)}| > r_k \end{cases}$$

(cf. Ref. 5). Then,

$$\int_{\mathbb{R}^2} |\nabla w_k(x)|^2 dx = \frac{2\pi}{\ln(1/r_k)}.$$

Furthermore,

$$\begin{aligned} \int_{\mathbb{R}^2} V(x)|w_k(x)|^2 d\mu(x) &\geq \int_{B(x^{(k)}, r_k^2)} V(x) d\mu(x) = t_k \mu(B(x^{(k)}, r_k^2)) \\ &\geq t_k c_0 r_k^{2\alpha} = c_0 \frac{A_{19}}{\ln \frac{1}{r_k}}. \end{aligned}$$

Hence, for any  $A_{19} > \frac{2\pi}{c_0}$ ,

$$\mathcal{E}_{V, \mu, \mathbb{R}^2}[w_k] < 0, \quad \forall k \in \mathbb{N},$$

and  $N_-(\mathcal{E}_{V, \mu, \mathbb{R}^2}) = \infty$ . □

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### APPENDIX: PROOFS OF (49), (50), AND (51)

Let  $\mathcal{B}(s) = (1+s)\ln(1+s) - s = \frac{1}{t}$ , then  $s = \mathcal{B}^{-1}(\frac{1}{t})$ . For small values of  $s$  (large values of  $t$ ), using

$$\ln(1+s) = s - \frac{s^2}{2} + \frac{s^3}{3} + O(s^4)$$

we have

$$(1+s)\ln(1+s) - s = \frac{s^2}{2} + O(s^3) = \frac{1}{t}.$$

One can write this in the form

$$\begin{aligned} \frac{s^2}{2} + s^2 g(s) &= \frac{1}{t}, \quad g(0) = 0, \\ \frac{s^2}{2} (1 + 2g(s)) &= \frac{1}{t}, \\ s(1 + h(s)) &= \sqrt{\frac{2}{t}}, \quad h(0) = 0, \end{aligned}$$

where  $g$  and  $h$  are  $C^\infty$  smooth functions in a neighborhood of 0. Let  $f(s) = s(1 + h(s))$ . Then,  $f(0) = 0$ ,  $f'(0) = 1$ , and  $(f^{-1})'(0) = 1$ , which means that both  $f$  and  $f^{-1}$  are invertible in a neighborhood of 0, and

$$s = f^{-1}\left(\sqrt{\frac{2}{t}}\right) = \sqrt{\frac{2}{t}} + O\left(\frac{1}{t}\right).$$

Thus,

$$\mathcal{B}^{-1}\left(\frac{1}{t}\right) = \sqrt{\frac{2}{t}}(1 + o(1)) \text{ as } t \rightarrow \infty$$

and

$$t\mathcal{B}^{-1}\left(\frac{1}{t}\right) = \sqrt{2t}(1 + o(1)) \text{ as } t \rightarrow \infty. \quad (\text{A1})$$

For large values of  $s$  (small values of  $t$ ), let  $\rho = 1 + s$  and  $r = \frac{1}{t}$ , then

$$\rho \ln \rho - \rho + 1 = r.$$

Let  $\rho = e^z$ , then

$$ze^z - r - e^z + 1 = 0. \quad (\text{A2})$$

This implies

$$\begin{aligned} (z-1)e^z &= r-1, \\ (z-1)e^{z-1} &= \frac{r-1}{e}. \end{aligned}$$

Let  $w := z-1$   $v := \frac{r-1}{e}$ . Then,

$$we^w = v. \quad (\text{A3})$$

The solution of (A3) is given by

$$w = \ln v - \ln \ln v + \frac{\ln \ln v}{\ln v} + O\left(\left(\frac{\ln \ln v}{\ln v}\right)^2\right)$$

[see (2.4.10) and the formula following (2.4.3) in Ref. 44]. So,

$$\begin{aligned} z &= 1 + \ln \frac{r-1}{e} - \ln \ln \frac{r-1}{e} + \frac{\ln \ln \frac{r-1}{e}}{\ln \frac{r-1}{e}} \\ &\quad + O\left(\left(\frac{\ln \ln \frac{r-1}{e}}{\ln \frac{r-1}{e}}\right)^2\right). \end{aligned}$$

Since

$$\begin{aligned} \ln(r-1) &= \ln r + O\left(\frac{1}{r}\right), \\ \ln(\ln(r-1)-1) &= \ln \ln r + O\left(\frac{1}{\ln r}\right), \end{aligned}$$

we get

$$\begin{aligned} z &= \ln r - \ln \ln r + \frac{\ln \ln r}{\ln r} + O\left(\frac{1}{\ln r}\right) \\ &= \ln \frac{1}{t} - \ln \ln \frac{1}{t} + \frac{\ln \ln \frac{1}{t}}{\ln \frac{1}{t}} + O\left(\frac{1}{\ln \frac{1}{t}}\right). \end{aligned}$$

This implies

$$\rho = e^z = \frac{1}{t \ln \frac{1}{t}} \left(1 + \frac{\ln \ln \frac{1}{t}}{\ln \frac{1}{t}} + O\left(\frac{1}{\ln \frac{1}{t}}\right)\right).$$

Hence,

$$t\mathcal{B}^{-1}\left(\frac{1}{t}\right) = \frac{1}{\ln \frac{1}{t}} \left(1 + \frac{\ln \ln \frac{1}{t}}{\ln \frac{1}{t}} + O\left(\frac{1}{\ln \frac{1}{t}}\right)\right),$$

implying

$$t\mathcal{B}^{-1}\left(\frac{1}{t}\right) = \frac{1}{\ln \frac{1}{t}}(1 + o(1)) \text{ as } t \rightarrow 0. \quad (\text{A4})$$

Let

$$\tau := t\mathcal{B}^{-1}\left(\frac{1}{t}\right). \quad (\text{A5})$$

Then,

$$\ln \frac{1}{t} = \frac{1 + o(1)}{\tau}. \quad (\text{A6})$$

From

$$\tau = \frac{1}{\ln \frac{1}{t}} \left( 1 + \frac{\ln \ln \frac{1}{t}}{\ln \frac{1}{t}} + O\left(\frac{1}{\ln \frac{1}{t}}\right) \right),$$

we get

$$\ln \frac{1}{t} = \frac{1 + \frac{\ln \ln \frac{1}{t}}{\ln \frac{1}{t}} + O\left(\frac{1}{\ln \frac{1}{t}}\right)}{\tau}. \quad (\text{A7})$$

Now, (A6) implies

$$\begin{aligned} \ln \frac{1}{t} &= \frac{1 + \frac{\ln \frac{1+o(1)}{1+o(1)} \tau + O\left(\frac{\tau}{1+o(1)}\right)}{\tau}}{\tau} \\ &= \frac{1 + (1 + o(1))\tau \ln \frac{1}{\tau} + O(\tau)}{\tau}. \end{aligned}$$

Substituting this into (A7), one gets

$$\begin{aligned} \ln \frac{1}{t} &= \frac{1 + \frac{\ln \frac{1+(1+o(1))\tau \ln \frac{1}{\tau} + O(\tau)}{1+(1+o(1))\tau \ln \frac{1}{\tau} + O(\tau)} \tau + O(\tau)}{\tau} \\ &= \frac{1}{\tau} - \ln \tau + O(1). \end{aligned}$$

Hence,

$$t = \tau e^{-\frac{1}{\tau}} e^{O(1)} \text{ as } \tau \rightarrow 0. \quad (\text{A8})$$

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