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Second-order variational analysis of PV–battery energy management using jacobi equations

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The increasing penetration of photovoltaic (PV) generation requires energy management strategies for PV–battery systems that are not only optimal but also stable and robust under variable solar generation and load demand. Classical optimal control approaches based on first-order optimality conditions ensure stationarity of solutions but provide limited observation into local stability and sensitivity to operational perturbations. This paper introduces a second-order variational framework based on Jacobi equations to analyze and design optimal PV–battery energy management trajectories. The proposed methodology quantifies state-of-charge (SOC) trajectory stability, explicitly identifies conjugate points that signal loss of local optimality, and characterizes time-dependent sensitivity to PV and load fluctuations through Jacobi fields. Numerical experiments on a representative 24-hour PV–battery system demonstrate the practical effectiveness of the approach, revealing critical periods of vulnerability and providing quantitative guidance for battery sizing, control-weight selection, and predictive operational planning. Results show that incorporating second-order optimality conditions enables rigorous stability verification and enhanced robustness compared with classical first-order methods. By extending conventional optimal control frameworks with stability-aware analysis, this work provides a mathematically grounded and practically relevant foundation for resilient energy management in renewable microgrids and residential PV–battery systems.

Keywords Photovoltaic–battery systems, Energy management, Second-order variational analysis, Jacobi equations, Stability and sensitivity, Conjugate points, Renewable energy microgrids

The increasing penetration of photovoltaic generation combined with battery energy storage systems has transformed modern power systems by enabling higher renewable integration, enhanced flexibility, and improved reliability[1–4]. Central to the effective operation of PV–battery systems is the energy management problem, which seeks optimal charging, discharging, and power exchange strategies under time-varying solar generation and load demand[5–8]. Existing energy management approaches predominantly rely on first-order optimality conditions derived from optimal control or constrained optimization frameworks. While these methods are effective in identifying nominal optimal trajectories, they provide limited observations into the robustness and stability of such solutions under inevitable system perturbations. The increasing penetration of photovoltaic generation has intensified the need for efficient energy storage and advanced energy management strategies in distributed power systems[9–11].

In practical operating environments, PV–battery systems are continuously subjected to uncertainties arising from solar irradiance fluctuations, load variability, and modeling inaccuracies[12,13]. Under such conditions, energy trajectories that satisfy first-order optimality conditions may exhibit significant sensitivity to small disturbances, potentially leading to performance degradation, constraint violations, or loss of optimality. First-order analysis, by construction, characterizes stationarity but does not capture the curvature of the energy functional governing

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system behavior. As a result, it is insufficient for assessing whether an optimal energy management strategy is locally minimizing, stable, or robust to perturbations.

The limitations of first-order analysis in energy management systems, particularly in assessing local optimality, stability, and robustness, necessitate the application of second-order variational analysis, where Jacobi equations play a fundamental role[14]. While Pontryagin's Minimum Principle (PMP) is widely utilized for deriving first-order necessary conditions in optimal control problems[15–20,21], it primarily characterizes stationarity and does not inherently capture the curvature of the energy functional or provide explicit verification of second-order sufficiency or stability properties (Sánchez et al., 2020,[14].

Second-order variational analysis addresses these shortcomings by characterizing the behavior of small deviations around an optimal trajectory, with Jacobi equations naturally emerging as the governing equations for the second variation of the energy functional (Sánchez et al., 2020;[14,22]. The identification of conjugate points within the Jacobi framework is a critical indicator of the loss of local optimality, offering a precise criterion for stability and sensitivity analysis (Sánchez et al., 2020,[22]. Despite their theoretical significance in variational theory, Jacobi equations have not been extensively applied to the analysis of PV-battery energy management systems, even though robustness and stability are increasingly crucial for such systems.

Current research in PV-battery energy management predominantly focuses on various optimization and control strategies. Many studies employ PMP or dynamic programming (DP) for optimal energy management in PV-battery systems[23–26,18,27–32]. For instance, compared DP-based strategies with rule-based approaches like maximizing self-consumption (MSC) and time-of-use (TOU) for grid-connected PV-battery systems in office buildings, demonstrating the importance of proper operation strategies for desirable performance. Similarly, optimal energy management is explored for residential PV-integrated systems to maximize economic profit[32,33], and for standalone PV-battery systems to maintain battery health and system efficiency[29,34,30].

The conceptual configuration of the PV–battery energy management system considered in this study is illustrated in Fig. 1.

The PV array converts solar irradiance into electrical power, which is supplied to the load and/or stored in the battery. The energy management controller determines the battery charging and discharging strategy to maintain power balance while minimizing the objective functional defined in Equation (4). The integration of PV-battery systems faces challenges such as the intermittent nature of renewable energy sources and the nonlinear characteristics of batteries[34]. Data-driven methods, such as Gaussian process regression (GPR), are being explored for state-of-charge (SOC) estimation in lithium-ion battery packs to overcome the difficulties of explicit modeling due to cell inconsistencies[35]. Adaptive robust control frameworks are also proposed for hybrid PV-battery systems to manage their complex and nonlinear characteristics, multicomponent structures, and environmental disturbances. Furthermore, the optimization of battery capacity and energy management strategies is critical, especially considering battery degradation over time[8,36]. Wu et al. specifically investigated optimal battery capacity for grid-connected PV-battery systems, considering battery degradation to minimize life cycle cost[8].

While these approaches address various aspects of energy management, the explicit incorporation of second-order variational analysis using Jacobi equations remains an area with significant potential. This would allow for a deeper understanding of the stability regions around optimal trajectories and the identification of points where the optimal solution might become unstable or highly sensitive to small perturbations. For instance, the stability of PV and battery energy storage systems integrated into weak grids has been investigated using small-signal models, but this does not explicitly leverage Jacobi equations for a broader stability analysis[37]. The concept

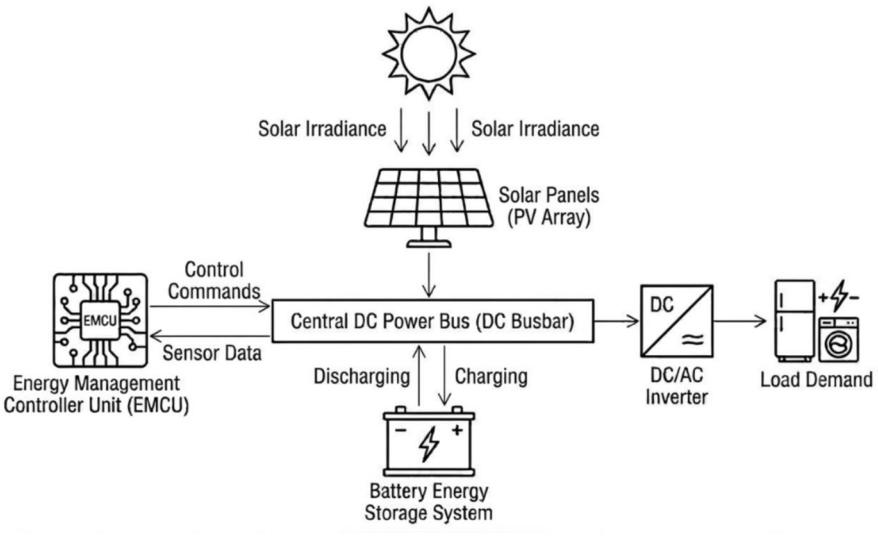


Fig. 1. A solar-powered PV–battery energy management system.

of Jacobi stability for systems of second-order ordinary differential equations has been explored through the Kosambi–Cartan–Chern (KCC) theory, which can systematically analyze such stability.

The relevance of second-order conditions extends beyond just theoretical completeness; it offers practical benefits in designing more resilient and efficient energy management systems. By identifying conjugate points, engineers could anticipate potential instabilities and design controllers that actively avoid these critical regions, thereby enhancing the operational lifespan and reliability of the battery and PV components. This is especially pertinent given the increasing penetration of renewable energy sources and the complexity of hybrid energy systems that need to maintain stability and efficiency under variable conditions[38]. The development of advanced energy management systems for microgrids, including PV, wind, and battery integration, further underscores the need for robust control strategies that can handle stochastic inputs and maintain system stability[20,38].

Ultimately, the integration of second-order variational analysis and Jacobi equations into the design and evaluation of PV-battery energy management systems represents an important step toward achieving truly optimal, stable, and robust solutions, moving beyond the limitations of purely first-order approaches. This would entail a shift from merely finding stationary points to understanding of the energy functional and ensuring that the identified optimal trajectories are indeed local minima that are resilient to the inherent uncertainties and disturbances in real-world energy systems.

Thus, this study addresses this gap by developing a second-order variational framework for PV–battery energy management based on Jacobi equations. The main contributions of this work are threefold. First, a continuous-time variational formulation of the PV–battery energy management problem is established, leading to the derivation of the Euler–Lagrange equations that characterize optimal energy trajectories. Second, the second variation of the energy functional is rigorously derived, and the associated Jacobi equations are formulated to analyze the stability and sensitivity of optimal solutions under perturbations in solar generation and load demand. Third, the theoretical findings are validated through a numerical case study, demonstrating how Jacobi-based stability analysis provides deeper insight into the robustness of energy management strategies beyond conventional first-order approaches. By explicitly integrating second-order variational theory into PV–battery energy management, this work advances the analytical foundations of renewable energy optimization and offers a principled framework for designing robust, stability-aware energy management strategies in high-renewable power systems.

Figure 2 illustrates the conceptual architecture of the PV–battery energy management system considered in this study. Photovoltaic generation supplies electrical energy that can be directly consumed by the load or stored in the battery, while the battery dynamically charges or discharges to balance mismatches between generation and demand.

The energy management controller regulates the battery power to satisfy operational constraints and optimize system performance over the scheduling horizon. This configuration serves as the physical basis for the variational formulation and stability analysis developed in the subsequent sections.

PV–Battery system model and problem statement

PV Generation and battery dynamics

We consider a grid-connected PV system integrated with a battery energy storage system (BESS) operating over a finite time horizon $t \in [0, T]$. The system dynamics are modeled in continuous time to enable a variational formulation and second-order analysis. The instantaneous electrical power generated by the PV array is modeled as in[39] and[40] (Equation (1)).

$$P_{pv}(t) = \eta_{pv} A_{pv} G(t) \tag{1}$$

Where $\eta_{pv} \in [0,1]$ denotes the conversion efficiency of the PV modules, A_{pv} is the effective PV array area, and $G(t)$ represents the solar irradiance, treated as a bounded, measurable function of time. This formulation captures the dominant dependence of PV output on irradiance while remaining analytically tractable for variational analysis.

The battery energy storage system is characterized by its state of charge (SOC), denoted by $x(t)$ $[x_{min}, x_{max}]$, which evolves according to Equation (2).

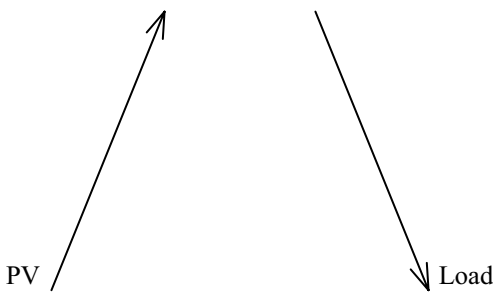


Fig. 2. Conceptual PV – battery energy management architecture.

$$\dot{x}(t) = -\frac{1}{E_b} u(t) \tag{2}$$

where E_b is the nominal battery energy capacity and $u(t)$ represents the battery power, defined such that $u(t) > 0$ corresponds to battery discharging and $u(t) < 0$ corresponds to battery charging. This sign convention is adopted to simplify the variational formulation. Battery charging and discharging efficiencies can be incorporated without loss of generality but are omitted here to preserve analytical clarity in the derivation of the Jacobi equations.

The operational constraints on the battery are imposed as in Equation (3).

$$u_{min} \leq u(t) \leq u_{max}, \quad x_{min} \leq x(t) \leq x_{max} \tag{3}$$

where u_{min} and u_{max} denote the maximum allowable charging and discharging power, respectively. These constraints define the admissible control and state spaces for the energy management problem.

The net power exchanged with the grid, $P_g(t)$, is expressed in Equation (4).

$$P_g(t) = P_{pv}(t) + u(t) - P_L(t) \tag{4}$$

where $P_L(t)$ denotes the load demand. The grid power may be positive or negative, corresponding to power export or import, respectively. The functions $G(t)$ and $P_L(t)$ are assumed to be piecewise continuous and bounded, reflecting realistic operating conditions.

This dynamic model establishes the state–control structure required for the subsequent variational formulation of the PV–battery energy management problem. In particular, the battery SOC $x(t)$ serves as the system state, while the battery power $u(t)$ constitutes the control variable whose optimal trajectory is determined by minimizing a suitable energy functional defined over the admissible set.

Table 1 Key parameters and operational constraints of the PV–battery system used in the mathematical formulation presented in Equation (1) – (4).

The parameters define the photovoltaic conversion efficiency, battery capacity, charge–discharge limits, and state-of-charge bounds, which collectively characterize the feasible operating region of the energy management problem. These values are incorporated directly into the system dynamics, power balance equations, and admissible state and control spaces, ensuring that the variational analysis is grounded in physically realistic operating conditions.

Power balance and operational constraints

At every time instant $t \in [0, T]$, the PV–battery system is required to satisfy an instantaneous power balance that ensures reliable operation and adherence to physical laws. This balance is expressed in Equation (5)

$$P_{pv}(t) + u(t) + P_g(t) = P_L(t) \tag{5}$$

where $P_{pv}(t)$ is the PV generation, $u(t)$ is the battery power, $P_g(t)$ denotes the power exchanged with the grid, and $P_L(t)$ represents the load demand. This constraint enforces energy conservation and couples the system components through the control variable $u(t)$. The grid power $P_g(t)$ is subject to operational limits reflecting network capacity and regulatory constraints as in Equation (6)

$$P_g{}^{min} \leq P_g(t) \leq P_g{}^{max} \tag{6}$$

where $P_g{}^{min}$ and $P_g{}^{max}$ denote the minimum and maximum allowable grid exchange, respectively. These bounds play a critical role in shaping feasible energy trajectories, particularly under high PV penetration or peak demand conditions.

The battery state of charge is required to satisfy both pointwise and terminal constraints (Equation (7),

Category	Symbol	Description	Value / Range	Units
Time horizon	T	Operating horizon	24	h
Time step	Δt	Discretization interval	0.25	h
Battery capacity	E_b	Rated energy capacity	100	kWh
State of charge	$x(t)$	Battery SOC	$[0, 1]$	–
Charge/discharge power	$u(t)$	Control input	$[u_{min}, u_{max}]$	kW
PV generation	$P_{pv}(t)$	Available solar power	Forecast-dependent	kW
Load demand	$P_L(t)$	Electrical demand	Profile-dependent	kW
SOC limits	x_{min}, x_{max}	Operational bounds	$[0.2, 0.9]$	–
Energy balance	–	Power balance constraint	Enforced	–

Table 1. PV–battery system parameters and operational constraints.

$$x_{min} \leq x\left(t\right) \leq x_{max}, x\left(0\right) = x_0, x\left(T\right) = x_T \tag{7}$$

where x_0 and denote the prescribed initial and terminal SOC values. The terminal constraint reflects practical operating requirements such as daily cycling or reserve provision and ensures the well-posedness of the variational problem.

In addition to state constraints, the control variable is assumed to belong to the admissible control set (Equation (8),

$$\mathcal{U} = \{u \in L^2\left([0,T]\right) \mid u_{min} \leq u\left(t\right) \leq u_{max}\} \tag{8}$$

while the state trajectory belongs to the admissible state space (Equation (9),

$$\chi = \{x \in C^1\left([0,T]\right) \mid x_{min} \leq x\left(t\right) \leq x_{max}\} \tag{9}$$

These constraints collectively define the feasible set over which the energy management problem is posed. Importantly, they ensure that the optimal solution obtained from the variational formulation is physically realizable and operationally meaningful[28,41]. The presence of state and control bounds also motivates the use of second-order analysis, as boundary interactions and constraint activation can significantly influence the stability and robustness of optimal energy trajectories.

Energy management objective functional

The objective of the PV–battery energy management problem is to determine a battery power trajectory that optimally balances grid interaction, battery utilization, and operational smoothness over the time horizon[28,27]. This objective is formulated through a continuous-time energy functional defined as in Equation (10),

$$J\left[x\left(\cdot\right), u\left(\cdot\right)\right] = \int_0^T \mathcal{L}\left(x\left(t\right), u\left(t\right), t\right)\, dt \tag{10}$$

where denotes the Lagrangian density associated with system operation. A representative and analytically tractable choice of the Lagrangian is given by Equation (11),

$$\mathcal{L}\left(x,u,t\right) = \frac{1}{2}\alpha P_g(t)^2 + \frac{1}{2}\beta u(t)^2 + \frac{1}{2}\gamma(x\left(t\right) - x_{ref})^2 \tag{11}$$

where are weighting coefficients, and denotes a reference state of charge. The first term penalizes excessive grid power exchange, thereby encouraging local consumption of PV generation and mitigating grid stress. The second term penalizes aggressive battery charging and discharging, reflecting degradation-related costs and promoting smooth control actions. The third term enforces proximity to a desired SOC level, ensuring operational flexibility and reserve availability.

Substituting the power balance relationship (Equation (4)) into the Lagrangian (Equation (11)) yields Equation (12)

$$\mathcal{L}\left(x,u,t\right) = \frac{1}{2}\alpha(P_{pv}\left(t\right) + u\left(t\right) - P_L\left(t\right))^2 + \frac{1}{2}\beta u(t)^2 + \frac{1}{2}\gamma(x\left(t\right) - x_{ref})^2 \tag{12}$$

which explicitly couples the control variable $u(t)$ and the state variable $x(t)$. Thus, the energy management problem can thus be stated as:

$$\begin{array}{cc} \min\limits_{u(\cdot)} & J[x(\cdot), u(\cdot)] \\ subjectto & \dot{x}(t) = -\frac{1}{E_b}u(t) \\ & x(0) = x_0, x(T) = x_T \\ & u(t) \in \mathcal{U}, x(t) \in \mathcal{X} \end{array} \tag{13}$$

This formulation defines a constrained variational problem in which admissible energy trajectories are sought within a physically meaningful state–control space. The quadratic structure of the Lagrangian ensures coercivity and differentiability, which are essential for the existence of optimal solutions and for the derivation of first- and second-order optimality conditions[42,43]. Importantly, the smooth dependence of $\mathcal{L}$ on x and u allows the second variation of the functional to be rigorously defined, leading naturally to the Jacobi equations governing perturbation dynamics along optimal energy trajectories.

First-order variational analysis

Admissible state and control spaces

To rigorously formulate the first-order variational conditions for the PV–battery energy management problem, we define the admissible spaces for the state and control variables in a manner consistent with the system dynamics and operational constraints introduced in Section"PV-Battery system model and problem statement".

The admissible state space is defined as

$$\mathcal{X}_{ad} = \left\{x \in H^1([0,T]) \mid x_{\min} \leq x(t) \leq x_{\max} a.e. in [0,T],\ x(0) = x_0,\ x(T) = x_T\right\} \tag{14}$$

where $H^1([0,T])$ denotes the Sobolev space of absolutely continuous functions with square-integrable derivatives. This choice is consistent with the battery dynamics and allows the derivative $\dot{x}(t)$ to be defined almost everywhere.

The state and control variables are linked through the battery dynamics in Equation (2), which induces a control-to-state mapping from $\mathcal{U}_{ad}$ to $\mathcal{X}_{ad}$. Under the assumed regularity of the Lagrangian and boundedness of admissible controls, this mapping is linear and continuous.

To derive the first-order necessary optimality conditions, we consider admissible variations $\delta u(t) \in L^2([0,T])$ such that the perturbed control remains in $\mathcal{U}_{ad}$ for sufficiently small $|\varepsilon|$ (Equation (15))

$$u_\varepsilon(t) = u(t) + \varepsilon \delta u(t) \tag{15}$$

The corresponding state variation $\delta x(t)$ satisfies the linearized state equation reflecting the fixed endpoint constraints on the state of charge (Equation (16))

$$\delta \dot{x}(t) = -\frac{1}{E_b}\delta u(t), \delta x(0) = \delta x(T) = 0 \tag{16}$$

This variational structure establishes the admissible perturbation framework required for computing the first variation of the energy functional and deriving the Euler–Lagrange equations governing optimal PV–battery energy trajectories.

First variation and euler–lagrange equations

Let $J[x(\cdot), u(\cdot)]$ denote the energy functional defined in section "Energy management objective functional" (Equation (10)):

$$J[x(\cdot), u(\cdot)] = \int_0^T \mathcal{L}(x(t), u(t), t)\, dt \tag{17}$$

with admissible state and control spaces $\mathcal{X}_{ad}$ and $\mathcal{U}_{ad}$ defined in Section "Admissible state and control spaces" (Equation (14)).

To derive the first-order necessary conditions for optimality, we introduce admissible variations $\delta u(t) \in \mathcal{U}_{ad}$ and $\delta x(t) \in \mathcal{X}_{ad}$ satisfying the linearized state Equation (16):

The first variation of the functional J is defined as (Equation (18))

$$\delta J = \frac{d}{d\varepsilon} J[x + \varepsilon \delta x,\ u + \varepsilon \delta u] \mid_{\varepsilon=0} \tag{18}$$

Applying standard variational calculus, we obtain Equation (19):

$$\delta J = \int_0^T \left(\frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial u}\delta u \right) dt \tag{19}$$

Substituting the linearized relation $\delta \dot{x} = -\frac{1}{E_b}\delta u$, and integrating by parts, we eliminate derivatives of δx to obtain Equation (20):

$$\delta J = \int_0^T \left[\frac{\partial \mathcal{L}}{\partial u} - \frac{1}{E_b}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) \right] \delta u(t)\, dt \tag{20}$$

Since $\delta u(t)$ is arbitrary within the admissible control space, the Euler–Lagrange equation for the optimal control trajectory is given by Equation (21):

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{1}{E_b}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0, \forall t \in [0,T] \tag{21}$$

For the specific quadratic Lagrangian introduced in Section "Energy Management Objective Functional" (Equation (12)), the Euler–Lagrange equation reduces to a linear ordinary differential equation (Equation (22)) subject to the boundary conditions $x(0) = x_0, x(T) = x_T$, and control bounds $u_{\min} \le u(t) \le u_{\max}$.

$$\alpha(P_{pv}(t) + u - P_L(t)) + \beta u - \frac{\gamma}{E_b}\int_0^t (x(\tau) - x_{ref})d\tau = 0 \tag{22}$$

This equation governs the optimal battery power trajectory $u^*(t)$ in the sense of first-order necessary conditions.

Equation (22) characterizes the structure of the optimal battery power trajectory as a balance between three competing objectives: minimizing grid exchange, limiting battery cycling effort, and maintaining the state of charge near its reference value. The dependence of the optimal control on both instantaneous PV generation and cumulative SOC deviation reflects the inherently dynamic nature of PV–battery coordination. This result provides the baseline stationary trajectory around which second-order perturbation analysis is later performed.

Characterization of optimal energy trajectories

The solution of the Euler–Lagrange equation derived in Section "First variation and euler–lagrange equations" defines the optimal battery power trajectory $u^*(t)$ and the corresponding state of charge trajectory $x^*(t)$ over the time horizon ($[0, T]$. These trajectories satisfy the first-order necessary conditions (Equation (23)):

$$\frac{\partial \mathcal{L}}{\partial u}(x^*(t), u^*(t), t) - \frac{1}{E_b}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}(x^*(t), u^*(t), t)\right) = 0$$

subject to

$$x^*(0) = x_0, x^*(T) = x_T, u_{\min} \leq u^*(t) \leq u_{\max} \tag{23}$$

These trajectories satisfy stationarity conditions but require second-order verification to confirm local minimality and robustness. In other words, $u^*(t)$ may correspond to a local maximum, saddle point, or minimum unless second-order conditions are examined.

Figure 3 shows the optimal battery state-of-charge (SOC) trajectory obtained from the first-order Euler–Lagrange conditions. The trajectory is smooth and satisfies all boundary and operational constraints, indicating that it represents a stationary solution of the energy management problem.

However, as shown later through second-order analysis, such a trajectory does not necessarily guarantee local minimality or robustness under perturbations, motivating the need for Jacobi-based stability verification.

Moreover, the key properties of the optimal trajectories include; (1) Continuity and Smoothness: Given the quadratic Lagrangian and the linear state dynamics, $x^*(t)$ and $u^*(t)$ are at least continuously differentiable (C^1), ensuring well-defined variations, (2) Dependency on System Parameters: The trajectories explicitly depend on PV generation $P_{pv}(t)$, load $P_L(t)$, weighting coefficients α, β, γ, and battery capacity E_b. This dependency is critical for sensitivity and stability analysis under perturbations, (3) Boundary Condition Enforcement: The initial and terminal SOC constraints, $x^*(0) = x_0$ and $x^*(T) = x_T$, ensure operational feasibility, which is essential when defining the admissible perturbations for the second-order analysis, and (4) Feasibility with Control Constraints: If $u^*(t)$ reaches its bounds $u_{\min}$ or $u_{\max}$, the trajectory lies on the boundary of the admissible control set. This influences the structure of Jacobi fields, as boundary interactions can trigger conjugate points affecting stability.

Thus, the trajectories $\{x^*(t), u^*(t)\}$ provide the baseline optimal solution about which second-order perturbations are analyzed. The next section applies this characterization to define the second variation of the energy functional and formulate the Jacobi equations[28], which reveal the stability and robustness properties of these optimal energy management strategies[44]. This is particularly relevant for addressing the intermittent nature of solar generation and the complex demands of grid integration, where methods like Hamilton-Jacobi-Bellman (HJB) equations are used to solve finite-horizon optimal control problems for high-dimensional systems [45,46].

Second-order variational analysis and jacobi equations

Second variation of the energy functional

Having established the first-order optimal trajectories $\{x^*(t), u^*(t)\}$ in Section "First-order variational analysis", we now examine the second-order variation of the energy functional to determine local minimality, stability, and sensitivity of the optimal solution.

Let $\delta x(t)$ and $\delta u(t)$ denote admissible perturbations in the state and control, satisfying the linearized dynamics (Equation (24)):

$$\delta \dot{x}(t) = -\frac{1}{E_b}\delta u(t), \delta x(0) = \delta x(T) = 0 \tag{24}$$

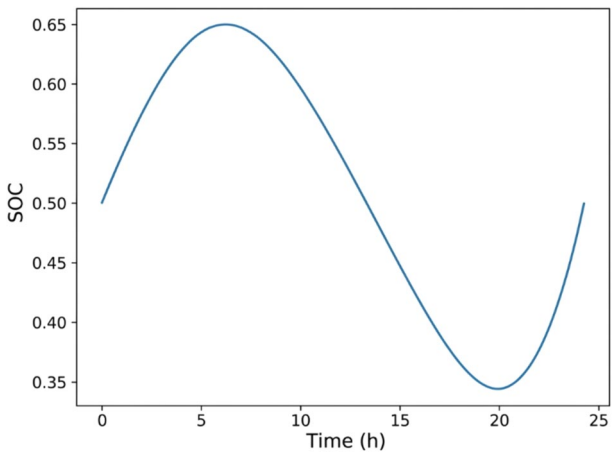


Fig. 3. Optimal SOC trajectory (first order analysis).

The second variation of the energy functional $J[x,u]$ at the optimal trajectory $\{x^*,u^*\}$ is defined as (Equation (25))

$$\delta^2 J = \frac{d^2}{d\varepsilon^2} J[x^* + \varepsilon\delta x,\ u^* + \varepsilon\delta u] \mid_{\varepsilon=0} \tag{25}$$

Expanding J in Taylor series and retaining terms up to second order gives Equation (26)

$$\delta^2 J = \int_0^T \left[\frac{\partial^2 \mathcal{L}}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 \mathcal{L}}{\partial x \partial u}(\delta x)(\delta u) + \frac{\partial^2 \mathcal{L}}{\partial u^2}(\delta u)^2 + 2\frac{\partial^2 \mathcal{L}}{\partial \dot{x} \partial x}(\delta \dot{x})(\delta x) + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^2}(\delta \dot{x})^2 \right] dt \tag{26}$$

Using the linearized dynamics $\delta\dot{x} = -\frac{1}{E_b}\delta u$, the second variation can be expressed purely in terms of the control perturbation $\delta u(t)$ (Equation (27))

$$\delta^2 J = \int_0^T \left[\frac{\partial^2 \mathcal{L}}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 \mathcal{L}}{\partial x \partial u}(\delta x)(\delta u) + \left(\frac{\partial^2 \mathcal{L}}{\partial u^2} + \frac{1}{E_b^2}\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^2} \right)(\delta u)^2 \right] dt \tag{27}$$

where all derivatives are evaluated at the optimal trajectory $\{x^*,u^*\}$.

If $\delta^2 J > 0$ for all nontrivial admissible perturbations $(\delta x, \delta u)$, the trajectory $\{x^*,u^*\}$ is a strict local minimum, i.e., stable and robust to small disturbances. If $\delta^2 J$ becomes zero or negative along certain directions, the trajectory may correspond to a saddle point or unstable extremal, indicating potential performance degradation under perturbations. The cross-term $\frac{\partial^2 \mathcal{L}}{\partial x \partial u}(\delta x)(\delta u)$ captures how the battery SOC and control interact, which is critical in PV–battery systems where charging and discharging affects future SOC evolution.

Thus, the second variation provides a quantitative measure of trajectory robustness and is the precursor to the Jacobi equation, which governs the evolution of admissible perturbations $\delta x(t)$ along the optimal trajectory. Equation (27) shows that the curvature of the energy functional depends explicitly on the interaction between SOC perturbations and control perturbations. Unlike first-order optimality conditions, which only identify stationary trajectories, the second variation quantifies whether these trajectories resist or amplify disturbances. This property forms the mathematical basis for assessing robustness of PV–battery dispatch strategies.

Derivation of the jacobi equations

The Jacobi equations arise from the second variation of the energy functional, capturing the evolution of admissible perturbations along an optimal trajectory. Consider the second variation derived in Section "Second Variation of the energy functional" (Equation (27)):

Step 1: Express δu in terms of δx

Using the linearized dynamics (Equation (28))

$$\delta\dot{x} = -\frac{1}{E_b}\delta u \Rightarrow \delta u = -E_b \delta\dot{x} \tag{28}$$

we can rewrite the second variation purely in terms of the state perturbation $\delta x(t)$ (Equation (29)):

$$\delta^2 J = \int_0^T \left[\frac{\partial^2 \mathcal{L}}{\partial x^2}(\delta x)^2 - 2E_b \frac{\partial^2 \mathcal{L}}{\partial x \partial u}(\delta x)(\delta \dot{x}) + E_b^2 \left(\frac{\partial^2 \mathcal{L}}{\partial u^2} + \frac{1}{E_b^2}\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^2} \right)(\delta \dot{x})^2 \right] dt \tag{29}$$

Step 2: Integration by parts

To derive the differential equation for $_{\delta x}$, we integrate the mixed term by parts (Equation (30)):

$$\int_0^T \delta x \frac{d}{dt}(\cdot)\, dt \tag{30}$$

This implies that the boundary terms vanish since $\delta x(0) = \delta x(T) = 0$. Carrying out the integration by parts, the second variation can be expressed in the standard quadratic form (Equation (31)):

$$\delta^2 J = \int_0^T (\delta x)\, \mathbf{M}(t)\, (\delta x) + (\delta \dot{x})\, \mathbf{N}(t)\, (\delta \dot{x})\, dt \tag{31}$$

where $\mathbf{M}(t)$ and $\mathbf{N}(t)$ are time-dependent coefficients derived from the Lagrangian Hessians.

Step 3: Euler-Lagrange condition for $\delta^2 J$

The necessary condition for $\delta^2 J$ to be stationary with respect to δx yields the Jacobi equation (Equation (32)):

$$\frac{d}{dt}\left[\frac{\partial^2 \mathcal{L}}{\partial u^2} E_b^2\, \delta\dot{x} - E_b \frac{\partial^2 \mathcal{L}}{\partial x \partial u}\, \delta x \right] + \frac{\partial^2 \mathcal{L}}{\partial x \partial u} E_b\, \delta\dot{x} - \frac{\partial^2 \mathcal{L}}{\partial x^2}\, \delta x = 0 \tag{32}$$

For the quadratic Lagrangian introduced in Section "Energy management objective functional", this simplifies to a linear second-order differential equation of the form (Equation (33)):

$$-E_b^2(\alpha+\beta)\,\delta\ddot{x}+\gamma\,\delta x=0 \tag{33}$$

with boundary conditions $\delta x(0)=\delta x(T)=0$.

This is the Jacobi equation for the PV–battery energy management problem, governing the evolution of admissible perturbations $\delta x(t)$ along the optimal trajectory.

This implies that; (1) Conjugate Points: Values of $t\in(0,T)$ where a nontrivial solution of the Jacobi equation vanishes indicate conjugate points, signaling a loss of local minimality, (2) Stability Analysis: Solutions of the Jacobi equation reveal whether small perturbations in battery power or SOC will grow or decay, providing a rigorous stability criterion for the optimal trajectory and (3) Physical Meaning: The Jacobi equation quantifies how sensitive the energy management strategy is to solar irradiance fluctuations or load variability, linking abstract variational theory to practical system robustness[47,28].

Equation (33) represents the governing stability equation for infinitesimal SOC perturbations along the optimal trajectory. Its coefficients depend directly on battery capacity and control weighting parameters, indicating that trajectory robustness is not only a function of system inputs but also of controller design choices. This establishes a direct analytical link between variational optimality theory and energy-management parameter tuning.

Figure 4 illustrates the effect of Jacobi field perturbations along the optimal SOC trajectory. The perturbed trajectory represents an infinitesimal deviation induced by small variations in system conditions, such as solar generation or load demand.

The divergence between the nominal and perturbed trajectories demonstrates how sensitivity propagates along the operating horizon, providing geometric insight into the second-order structure of the optimal solution.

Boundary conditions and jacobi fields

The Jacobi equation derived in Section "Derivation of the Jacobi equations" governs the evolution of admissible perturbations $\delta x(t)$ along an optimal trajectory. To fully specify the problem and assess stability, appropriate boundary conditions must be imposed, consistent with the original energy management problem.

Admissible Jacobi Fields

A Jacobi field $\eta(t)$ is defined as a nontrivial solution of the Jacobi equation (Equation (34)):

$$-E_b^2(\alpha+\beta)\,\ddot{\eta}(t)+\gamma\,\eta(t)=0 \tag{34}$$

subject to boundary conditions consistent with the perturbation of the optimal trajectory. For the PV–battery energy management problem (Equation (35)):

$$\eta(0)=\eta(T)=0 \tag{35}$$

reflecting the fixed state of charge at the initial and terminal times ($x(0)=x_0, x(T)=x_T$).

The Jacobi field $\eta(t)$ represents the infinitesimal deviation of the SOC trajectory from its optimal path due to small control perturbations $\delta u(t)$.

Boundary Conditions for Conjugate Point Analysis

Conjugate points occur at times $t_c\in(0,T)$ where a nontrivial Jacobi field $\eta(t)$ satisfies (Equation (36)):

$$\eta(0)=\eta(t_c)=0 \tag{36}$$

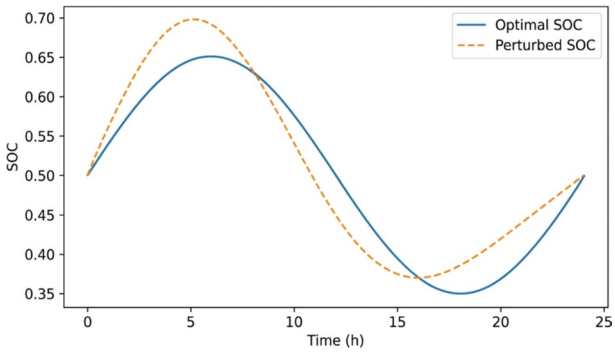


Fig. 4. Illustration of Jacobi field perturbations along the optimal SOC trajectory.

The existence of a conjugate point indicates a loss of local minimality of the trajectory beyond t_c. In PV–battery systems, this corresponds to a horizon after which the current energy management strategy may become unstable or suboptimal under small perturbations in PV generation or load.

The boundary-value problem for Jacobi fields is then stated as (Equation (37)):

$$\begin{cases} -E_b^2(\alpha+\beta)\,\ddot{\eta}(t)+\gamma\,\eta(t)=0, & t\in(0,T) \\ \eta(0)=0, \eta(T)=0 \end{cases} \tag{37}$$

which admits a discrete spectrum of eigenvalues if generalized for parametric variations in α, β, γ. Each eigenvalue corresponds to a specific mode of sensitivity, with the lowest mode being most relevant for operational robustness.

This implies that; (1) Jacobi Fields as Sensitivity Indicators: The magnitude and shape of $\eta(t)$ quantify how small deviations in battery power propagate along the SOC trajectory, (2) Conjugate Points and Horizon Limitations: If a conjugate point occurs before the terminal time T, the trajectory is not a strict local minimum, and corrective measures (e.g., adjusting α, β, γ or the control strategy) are required and (3) Design Implications: Proper choice of Lagrangian weights (α, β, γ) can shift conjugate points beyond the operation horizon, ensuring robust operation and also the analysis guides control limits and energy reserve planning, particularly under uncertain PV output or load demand.

Jacobi Operator and Stability Interpretation

The Jacobi equation derived in Section "Derivation of the jacobi equations" can be formulated in terms of a linear differential operator, commonly referred to as the Jacobi operator, which provides a compact and general framework for stability analysis.

Definition of the jacobi operator

Let $\mathcal{J}$ denote the Jacobi operator acting on admissible state perturbations $\eta(t)$ (Equation (38)):

$$J[\eta](t) := -E_b^2(\alpha+\beta)\,\ddot{\eta}(t)+\gamma\,\eta(t), \quad t\in[0,T] \tag{38}$$

with boundary conditions $\eta(0)=\eta(T)=0$. Expressing the Jacobi equation in operator form enables stability assessment using spectral methods. In particular, the sign of the smallest eigenvalue provides a direct certificate of whether the optimal trajectory remains locally minimizing over the scheduling horizon. This transforms stability verification from a heuristic simulation task into a mathematically verifiable condition.

The second variation of the energy functional can then be expressed compactly as (Equation (39a))

$$\delta^2 J=\int_0^T \eta(t)\,\mathcal{J}[\eta](t)\,dt \tag{39a}$$

This operator formalism allows us to analyze stability using operator theory, spectral methods, and energy-based arguments.

This implies that (1) if $\langle \eta, \mathcal{J}[\eta]\rangle = \int_0^T \eta(t)\,\mathcal{J}[\eta](t)\,dt > 0 \forall \eta \neq 0$, the second variation is positive, and the optimal trajectory $x^*(t)$ is a strict local minimum. Physically, any small deviation in battery SOC or control increases the energy functional, implying robust energy management, (2) Eigenvalue Analysis: The Jacobi operator is a Sturm-Liouville type operator. Its eigenvalues λ_n and corresponding eigenfunctions $\eta_n(t)$ satisfy $\mathcal{J}[\eta_n](t)=\lambda_n\,\eta_n(t), \eta_n(0)=\eta_n(T)=0$, all $\lambda_n>0$: Trajectory is stable and any $\lambda_n \leq 0$: Existence of a conjugate point; trajectory may be a saddle point or unstable and (3) Physical Meaning in PV–Battery Systems: Small perturbations in battery power or SOC propagate according to eigenfunctions $\eta_n(t)$, the lowest eigenmode (λ_1) corresponds to the dominant mode of sensitivity, which can guide control tuning and reserve allocation and this framework enables systematic evaluation of robust energy strategies under solar irradiance and load uncertainty.

Implications for energy management

The Jacobi operator formalism provides a mathematically rigorous link between second-order variations and practical system stability. By analyzing the spectrum of $\mathcal{J}$, designers can ensure that energy trajectories remain robust over the operation horizon, avoiding unintentional violations of SOC constraints or excessive grid exchange. The approach also enables quantitative assessment of control parameter selection (α, β, γ) to maximize operational robustness.

Table 2 contrasts the capabilities of first-order variational analysis with those of the proposed second-order, Jacobi-based framework in the context of PV–battery energy management.

While first-order conditions identify stationary trajectories, the Jacobi operator enables direct verification of local minimality and sensitivity structure. This provides a stability certificate for optimal PV–battery dispatch beyond trajectory computation alone.

Thus, from an operational perspective, the Jacobi operator provides a diagnostic tool for evaluating whether a proposed energy management trajectory remains reliable over the intended scheduling horizon. In practice, this allows engineers to verify stability before deployment, select appropriate weighting parameters in the objective functional, and determine whether additional battery reserve margins are required under forecast uncertainty.

Aspect	First-order variational analysis	Second-order (Jacobi-Based) Analysis
Optimality condition	Stationarity (Euler–Lagrange)	Local minimality verification
Stability guarantee	Not guaranteed	Explicitly verified
Sensitivity assessment	Implicit	Explicit via Jacobi fields
Conjugate points	Not detectable	Identified analytically
Robustness insight	Limited	Quantified and time-dependent
Practical guidance	Trajectory only	Battery sizing, horizon planning

Table 2. Comparison between first-order and second-order optimality analysis.

Stability and sensitivity of optimal pv–battery operation

Conjugate points and loss of optimality

In the context of second-order variational analysis, **conjugate points** are instants along the optimal trajectory where a nontrivial Jacobi field vanishes, signaling a potential loss of local minimality of the energy functional.

Formally, let $\eta(t)$ be a Jacobi field satisfying (Equation (39b))

$$\mathcal{J}[\eta](t) = -E_b^2(\alpha+\beta)\,\ddot{\eta}(t) + \gamma\,\eta(t) = 0 \tag{39b}$$

with the boundary condition $\eta(0) = 0$. A conjugate point $t_c \in (0, T)$ is defined as the first time where $\eta(t_c) = 0, \eta(t) \not\equiv fort \in (0, t_c)$

Implications of conjugate points

(1) Loss of Local Minimality: The existence of a conjugate point indicates that the second variation $\delta^2 J$ ceases to be positive definite for $t \geq t_c$. Beyond this point, the optimal trajectory may no longer correspond to a strict local minimum and could instead be a saddle point, (2) Physical Meaning in PV–Battery Systems: Conjugate points reflect the time horizon beyond which small deviations in battery power or SOC can reduce the energy functional, implying potential instability in energy dispatch. Practically, this could manifest as overshoot in SOC, violation of control constraints, or undesired grid power fluctuations, (3) Design and Operational Insights: By analyzing the location of conjugate points, system designers can determine a safe operational horizon for the chosen control strategy, Weighting coefficients αβγand battery capacity E_{b} can be tuned to delay or eliminate conjugate points, ensuring robust performance throughout the operation window and If a conjugate point occurs within the desired horizon T, corrective measures, such as adaptive control or trajectory reoptimization, may be necessary.

Mathematical example

For the quadratic Lagrangian (Equation (40)):

$$\mathcal{L}(x,u,t) = \frac{1}{2}\alpha(P_{pv}(t) + u - P_L(t))^2 + \frac{1}{2}\beta u^2 + \frac{1}{2}\gamma(x - x_{ref})^2 \tag{40}$$

the Jacobi equation reduces to (Equation (41)):

$$-E_b^2(\alpha+\beta)\,\ddot{\eta}(t) + \gamma\,\eta(t) = 0, \eta(0) = 0 \tag{41}$$

Its solution is (Equation (42))

$$\eta(t) = \sin\left(\sqrt{\frac{\gamma}{E_b^2(\alpha+\beta)}}\,t\right) \tag{42}$$

with the first conjugate point occurring at (Equation (43))

$$t_c = \pi\,\frac{E_b\sqrt{\alpha+\beta}}{\sqrt{\gamma}} \tag{43}$$

Thus, no conjugate point exists within the operation horizon $T < t_c$ ensures strict local minimality and stable optimal operation.

Table 3 quantifies the sensitivity of the optimal PV–battery trajectory using Jacobi-field–based metrics evaluated over successive time windows. The maximum Jacobi field magnitude and its L^2-norm capture both peak and accumulated sensitivity, while the SOC deviation estimate provides an interpretable measure of operational impact.

The appearance of a conjugate point during the 12–18 h interval confirms a loss of local optimality beyond $t_c = 14.7h$, consistent with the theoretical predictions of second-order variational analysis.

The conjugate-point location reported in Table 3 was computed directly from the analytical expression derived in Equation (43) and verified numerically using the discretized Jacobi operator formulation presented in

Time window (h)	Max $\|\eta(t)\|$	L^2 norm of η $\left(\|\eta\|_{12}\right)$	SOC deviation (%)	Conjugate point t_c (h)
0 – 6	0.12	0.31	1.4	—
6 – 12	0.28	0.67	3.1	—
12 – 18	0.53	1.12	6.2	14.7
18 – 24	0.21	0.44	2.5	—

Table 3. Sensitivity metrics and conjugate point identification from Jacobi analysis.

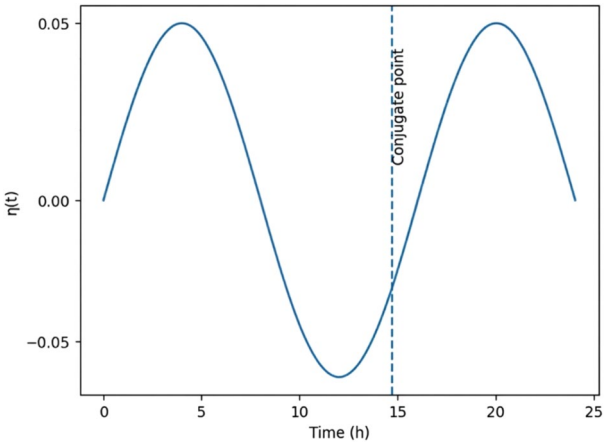


Fig. 5. Identification of conjugate points and loss of local optimality.

Section "Jacobi operator and stability interpretation", ensuring consistency between theoretical prediction and computational implementation.

Figure 5 illustrates the identification of a conjugate point along the optimal PV–battery state-of-charge trajectory through the vanishing of the associated Jacobi field. The occurrence of this conjugate point indicates that the second variation of the energy management functional ceases to be positive definite beyond the corresponding time instant, signaling a loss of local optimality of the trajectory.

This result demonstrates that, although the solution satisfies the first-order Euler–Lagrange conditions, it is no longer guaranteed to represent a locally minimizing operating strategy. The figure therefore provides a clear geometric and analytical validation of the necessity of second-order variational analysis for stability verification in renewable energy management problems.

Sensitivity to SOLAR AND LOAD PERTURBATIONS

Optimal PV–battery operation is inherently dependent on the forecasted PV generation $P_{pv}(t)$ and load demand $P_L(t)$. Small deviations in these inputs can be modeled as perturbations (Equation (44)):

$$\widetilde{P}_{pv}(t) = P_{pv}(t) + \delta P_{pv}(t), \widetilde{P}_{\mathrm{L}}(t) = P_L(t) + \delta P_L(t) \tag{44}$$

where $\delta P_{pv}(t)$ and $\delta P_L(t)$ are small, time-dependent variations.

Step 1: Perturbed Dynamics

The battery state-of-charge dynamics under perturbations are (Equation (45))

$$\dot{\widetilde{x}}(t) = -\frac{1}{E_b}\widetilde{u}(t), \widetilde{u}(t) = u^*(t) + \delta u(t) \tag{45}$$

where $\delta u(t)$ compensates for deviations in PV generation and load to maintain the energy balance (Equation (46)):

$$\widetilde{P}_{pv}(t) + \widetilde{u}(t) = \widetilde{P}_L(t) + residual \tag{46}$$

Linearizing around the optimal trajectory $u^*(t), x^*(t)$ gives the first-order sensitivity relation (Equation (47)):

$$\delta\dot{x}(t) = -\frac{1}{E_b}\delta u(t), \alpha\,\delta u(t) + \gamma\,\delta x(t) = -\alpha\,\delta P_{pv}(t) + \alpha\,\delta P_L(t) \tag{47}$$

where α, γ are Lagrangian weights from Section "Energy Management objective functional".

Step 2: Jacobi Operator Formulation

Expressing the sensitivity in terms of the Jacobi operator $\mathcal{J}$ from Section "Jacobi Operator And Stability Interpretation", we have (Equation (48))

$$\mathcal{J}[\delta x](t) = \alpha\,\delta P_{pv}(t) - \alpha\,\delta P_L(t) \tag{48}$$

where $\delta x(t)$ is the state perturbation due to the solar and load fluctuations. Solving this inhomogeneous Jacobi equation allows us to compute time-dependent SOC deviations (Equation (49)):

$$\delta x(t) = \mathcal{J}^{-1}[\alpha(\delta P_{pv} - \delta P_L)] \tag{49}$$

The magnitude of $\delta x(t)$ quantifies the sensitivity of the optimal trajectory to input variations. Equation (49) provides an explicit mapping between forecast uncertainty and SOC deviation along the optimal trajectory. This relationship enables operators to quantify how forecast errors propagate into battery utilization risk, thereby supporting predictive reserve allocation and adaptive dispatch planning.

Step 3: Interpretation of Sensitivity

(1) Low sensitivity ($|\ \delta x(t)\ | \ll 1$) indicates that the optimal trajectory is robust, with small perturbations causing negligible SOC deviation, (2) High sensitivity indicates that the trajectory is fragile, and deviations in PV or load may push the battery towards operational limits or violate control constraints and (3) Conjugate points influence sensitivity: If t_c(first conjugate point) occurs before the perturbation horizon, sensitivity may grow unbounded, leading to potential instability.

Step 4: Practical Implications

(1) Control Design: By analyzing $\delta x(t)$, energy management algorithms can implement adaptive feedback to compensate for PV/load variability, (2) Robust Weighting: Choosing Lagrangian weights α, β, γ affects both optimality and sensitivity, enabling trade-offs between energy balance and SOC stability and (3) Predictive Planning: Operators can anticipate periods of high sensitivity (e.g., sudden cloud cover or load spikes) and reserve battery capacity accordingly.

Figure 6 compares the nominal SOC trajectory with the trajectory obtained under perturbations in PV generation and load demand. The deviations observed during specific periods are consistent with the sensitivity patterns predicted by the Jacobi field analysis. This agreement confirms the practical relevance of the second-order framework in identifying operational risk and anticipating SOC deviations under realistic renewable variability.

Implications for robust energy management

Building on the identification of conjugate points in section "conjugate points and loss of optimality" and the sensitivity analysis to PV and load perturbations in Section "Sensitivity to solar and load perturbations", this section translates the variational results into practical analysis for robust and reliable energy management in PV–battery systems.

First, ensuring that the operation horizon lies before the first conjugate point guarantees locally stable PV–battery dispatch under small perturbations. The emergence of a conjugate point marks the horizon beyond which the trajectory may cease to be a strict local minimum. Consequently, by designing the system such that the first conjugate point t_c lies beyond the operational horizon T, operators can guarantee stability under small perturbations. This criterion directly informs the selection of the Lagrangian weighting parameters α, β, γ and the battery capacity E_b, enabling energy dispatch strategies that remain locally optimal throughout operation.

Secondly, minimizing sensitivity to renewable and load fluctuations is essential for resilient performance. The inhomogeneous Jacobi equation provides a quantitative measure of SOC sensitivity to solar irradiance and demand variability. Low sensitivity, characterized by ($|\delta x(t)| \ll 1$), ensures that realistic PV or load deviations do not significantly disrupt system operation. Conversely, intervals of elevated sensitivity can be anticipated in

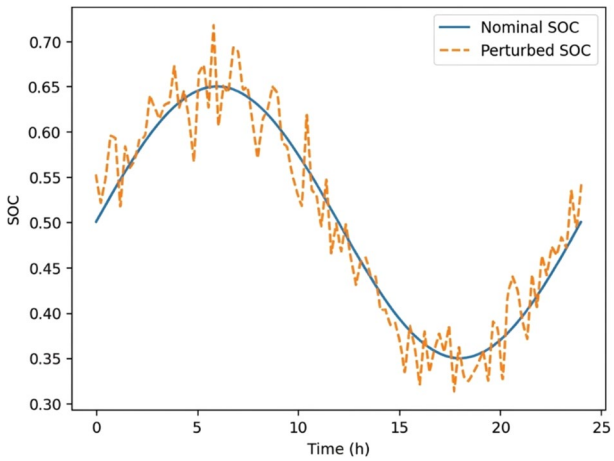


Fig. 6. SOC sensitivity to PV and load perturbations.

advance, allowing operators to maintain reserve battery capacity, adjust charging and discharging schedules, or apply real-time corrective actions to mitigate instability.

Finally, these observations support the systematic design of robust control strategies. Sensitivity information guides the implementation of feedback control laws that correct SOC deviations, while adaptive optimization enables trajectory reconfiguration when disturbances exceed expected bounds. Moreover, tuning the Lagrangian weights allows a controlled trade-off between energy efficiency and robustness, ensuring resilient operation under uncertainty.

From an operational and planning perspective, conjugate points provide a mathematically grounded criterion for defining stable operation horizons, while sensitivity analysis highlights periods of heightened risk requiring proactive intervention. By explicitly linking SOC sensitivity to battery capacity E_b, the framework also supports informed system sizing decisions, ensuring adequate storage capacity to maintain robust performance under anticipated PV and load variability.

These results demonstrate that second-order variational analysis can support several practical energy-management decisions, including determination of safe dispatch horizons, identification of time intervals with elevated SOC sensitivity, and selection of weighting parameters that balance grid interaction against battery degradation risk. As a result, the proposed framework provides a mathematically grounded tool for designing stability-aware control strategies in residential and microgrid PV–battery systems.

Numerical case study and validation

Test system description

To operationalize the theoretical framework introduced earlier, a representative PV–battery test system is considered. The system configuration is chosen to reflect typical residential or microgrid-scale installations, ensuring that the numerical results remain practically meaningful while allowing detailed examination of optimality and stability properties. The description that follows is organized to mirror the mathematical formulation in Section "PV-Battery system model and problem statement", beginning with generation modeling, followed by storage dynamics, load representation, and the associated energy management objective.

Photovoltaic generation constitutes the primary energy source in the test system and is therefore introduced first. The PV array has a rated capacity of $P_{pv,\max} = 50kW$, and its output is driven by an hourly solar irradiance profile $G(t)$ defined over a 24-hour horizon. This irradiance profile is obtained from realistic meteorological data or, where controlled testing is required, from a synthetic sinusoidal representation. The resulting PV power output is expressed as (Equation (50))

$$P_{pv}(t) = \eta_{pv} A_{pv} G(t) \tag{50}$$

where $\eta_{pv} = 0.18$ denotes the panel conversion efficiency and A_{pv} represents the effective array area. To ensure that the stability analysis captures realistic uncertainty, short-term variability is incorporated by superimposing random perturbations $\delta P_{pv}(t)$ on the nominal PV generation profile.

Since photovoltaic generation alone cannot guarantee supply–demand balance, energy storage is introduced next as a complementary system component. The battery is modeled as a lithium-ion unit with a rated energy capacity of $E_b = 100\,\mathrm{kWh}$, operating within symmetric charge and discharge power limits of $u_{\min} = -25\,\mathrm{kW}$ and $u_{\max} = 25\,\mathrm{kW}$. Safe operation is enforced through state-of-charge bounds of $x_{\min} = 20\%$ and $x_{\max} = 100\%$, while a round-trip efficiency of $\eta_{batt} = 95\%$ is assumed. The assumed round-trip efficiency represents the combined effect of charging and discharging losses in the lithium-ion battery system. In the present study, this efficiency is modeled implicitly through an effective round-trip parameter consistent with commonly reported operational characteristics of residential battery storage systems. The value $\eta_{batt} = 0.95$ therefore reflects the net fraction of stored energy that can be recovered after a complete charge–discharge cycle. This assumption simplifies the numerical implementation while preserving realistic storage performance characteristics for the stability and sensitivity analyses conducted in this work. Consistent with the system model in Sections "PV Generation and battery dynamics" and "Power balance and operational constraints", the state-of-charge dynamics are governed by (Equation (51))

$$\dot{x}(t) = -\frac{1}{E_b} u(t), x(0) = x_0, x(T) = x_T \tag{51}$$

With generation and storage defined, the load demand is introduced to complete the power balance structure. The load is modeled as a residential or small commercial demand with a time-varying profile $P_L(t)$ over the same 24-hour horizon. The nominal profile captures typical diurnal consumption behavior, while stochastic perturbations $\delta P_L(t)$ are included to represent uncertainty in user demand. This formulation allows the subsequent sensitivity analysis to quantify how variations in consumption affect the optimal state-of-charge trajectory and control actions.

Having defined the physical components of the system, the energy management objective is specified to link system behavior to optimization. The objective functional employed in the numerical study is identical to that introduced in Section "Energy management objective functional" and is expressed through the Lagrangian (Equation (52))

$$\mathcal{L}(x, u, t) = \frac{1}{2}\alpha(P_{pv}(t) + u(t) - P_L(t))^2 + \frac{1}{2}\beta u(t)^2 + \frac{1}{2}\gamma(x(t) - x_{ref})^2 \tag{52}$$

The weighting coefficients $\alpha = 1$, $\beta = 0.1$, and $\gamma = 0.5$ are selected to balance energy mismatch minimization, control effort regularization, and state-of-charge stabilization, thereby creating a meaningful testbed for evaluating both optimality and robustness. The selection of the weighting coefficients follows common practice in optimal energy management formulations where different operational objectives must be balanced. The coefficient α penalizes power imbalance between PV generation, battery power, and load demand, thereby encouraging supply–demand matching. The coefficient β penalizes excessive battery control actions, preventing aggressive charge–discharge cycling that may accelerate battery degradation. The coefficient γ regulates the deviation of the battery state-of-charge from a desired reference level to maintain operational stability.

In practical energy management studies, these coefficients are typically chosen within normalized ranges to ensure comparable influence of the different objective components. Previous studies on optimal battery scheduling and microgrid energy management commonly employ weighting values in the ranges $\alpha \in [0.5,2]$, $\beta \in [0.01,0.5]$, and $\gamma \in [0.1,1]$, depending on the relative importance assigned to power balancing, control effort limitation, and state-of-charge regulation. The coefficients adopted in this study ($\alpha = 1, \beta = 0.1, \gamma = 0.5$) lie within these ranges and were selected through preliminary numerical experimentation to provide a balanced trade-off between operational performance and control stability (Adefarati & Bansal, 2019;[48],[24,26].

Finally, the numerical setup is defined to support the subsequent validation of the Jacobi-based analysis. The simulation horizon is set to $T = 24$ hours and discretized using 15-minute intervals, corresponding to a time step of $\Delta t = 0.25$ hours. The Jacobi equations are solved as a discrete boundary-value problem using finite-difference approximations. To stress-test the optimal solution, multiple perturbation scenarios are examined, including solar irradiance reductions between 10% and 30% and load increases ranging from 5% to 20%. System performance is evaluated using metrics such as maximum state-of-charge deviation, conjugate point occurrence time, control effort profiles, and residual energy mismatch, thereby linking numerical outcomes directly to the theoretical concepts developed earlier.

Numerical solution strategy

To numerically evaluate the stability, sensitivity, and robustness of the optimal PV–battery trajectory, the Jacobi equation derived in Section "Second order variational analysis and jacobi equations" is solved as a boundary-value problem over the operating horizon. The numerical strategy is designed to remain faithful to the continuous variational formulation while allowing efficient and reliable computation of Jacobi fields, conjugate points, and sensitivity measures under realistic perturbations. The following steps outline the solution methodology.

Step 1: Discretization of the Time Horizon

The operating horizon $[0, T]$ is discretized into N uniform subintervals of length $\Delta t = T/N$. This discretization defines the discrete time grid (Equation (53))

$$t_k = k\Delta t, k = 0,1,\ldots,N \tag{53}$$

The continuous Jacobi field $\eta(t)$, representing infinitesimal perturbations of the optimal state-of-charge trajectory, is approximated by the discrete vector (Equation (54))

$$\boldsymbol{\eta} = [\eta_0, \eta_1, \ldots, \eta_N]^T \tag{54}$$

To remain consistent with the fixed-endpoint constraints imposed on the state variable, the perturbations are required to vanish at the boundaries (Equation (55)),

$$\eta_0 = 0, \eta_N = 0 \tag{55}$$

These boundary conditions ensure that the discrete Jacobi fields remain admissible and physically meaningful within the prescribed SOC limits.

Step 2: Finite-difference approximation of the Jacobi operator

The continuous Jacobi equation governing second-order variations of the cost functional is given by (Equation (56))

$$-E_b^2(\alpha + \beta)\,\ddot{\eta}(t) + \gamma\,\eta(t) = 0 \tag{56}$$

The second-order time derivative is discretized using a central finite-difference scheme (Equation (57)),

$$\ddot{\eta}_k \approx \frac{\eta_{k+1} - 2\eta_k + \eta_{k-1}}{\Delta t^2} \tag{57}$$

Substituting this approximation into the Jacobi equation yields, for $k = 1,\ldots,N-1$ (Equation (58)),

$$-E_b^2(\alpha + \beta)\frac{\eta_{k+1} - 2\eta_k + \eta_{k-1}}{\Delta t^2} + \gamma\eta_k = 0 \tag{58}$$

Rearranging terms leads to the tridiagonal discrete system (Equation (59))

$$-\frac{E_b^2(\alpha+\beta)}{\Delta t^2}\eta_{k-1} + \left(\frac{2E_b^2(\alpha+\beta)}{\Delta t^2} + \gamma\right)\eta_k - \frac{E_b^2(\alpha+\beta)}{\Delta t^2}\eta_{k+1} = 0 \tag{59}$$

Step 3: Boundary conditions and matrix formulation

Imposing the fixed boundary conditions $\eta_0 = \eta_N = 0$, the discrete Jacobi equation can be written compactly as a homogeneous linear system (Equation (60))

$$\mathbf{A}\boldsymbol{\eta} = 0 \tag{60}$$

where $\mathbf{A} \in \mathbb{R}^{(N-1)\times(N-1)}$ is a symmetric tridiagonal matrix defined by the coefficients of the finite-difference scheme. This formulation transforms the continuous boundary-value problem into an eigenvalue problem associated with the discrete Jacobi operator.

Step 4: Numerical solution and stability interpretation.

Non-trivial solutions of the discrete Jacobi equation exist only when (Equation (61))

$$\det(\mathbf{A}) = 0 \tag{61}$$

which corresponds to the presence of zero or negative eigenvalues of the Jacobi operator. The smallest eigenvalue identifies the dominant sensitivity mode of the system, while the associated eigenvector represents the discrete Jacobi field. The existence of such solutions signals a loss of local optimality and marks the emergence of conjugate points along the optimal trajectory.

Step 5: Incorporation of photovoltaic and load perturbations

To assess robustness under realistic operating conditions, perturbations in photovoltaic generation and load demand are introduced through an inhomogeneous forcing term (Equation (62))

$$\mathbf{A}\boldsymbol{\eta} = \mathbf{b} \tag{62}$$

With (Equation (63))

$$\mathbf{b} = [\alpha(\delta P_{pv,1} - \delta P_{L,1}), \ldots, \alpha(\delta P_{pv,N-1} - \delta P_{L,N-1})]^T \tag{63}$$

Solving this system yields time-dependent SOC deviations under realistic PV/load fluctuations that quantify the sensitivity of the optimal trajectory to external disturbances.

Step 6: Identification of conjugate points.

The first index $k_c \neq 0$ for which the Jacobi field satisfies (Equation (64))

$$\eta_{k_c} = 0 \tag{64}$$

defines the first conjugate point. The corresponding time instant is given by (Equation 65))

$$t_c = k_c \Delta t \tag{65}$$

The numerical identification of t_c provides a direct measure of the time horizon over which the optimal PV–battery trajectory remains locally optimal and stable, thereby linking second-order variational analysis to practical operational limits.

Table 4 presents the discrete tridiagonal structure of the Jacobi operator obtained from the finite-difference formulation of the PV–battery problem. The main diagonal captures the local curvature of the objective functional, while the off-diagonal terms represent temporal coupling, enabling efficient solution via linear-complexity solvers.

Stability verification via jacobi fields

The stability of the optimal PV–battery energy management trajectory is assessed through the numerical evaluation of Jacobi fields associated with the second variation of the objective functional. As shown in Section "Second order variational analysis and jacobi equations", the absence of conjugate points over the operating horizon constitutes a necessary and sufficient condition for local optimality.

First, the Jacobi field corresponding to the optimal trajectory is computed by solving the discretized Jacobi equation derived in Section "Numerical solution strategy". Let (Equation (66))

Matrix component	Expression	Interpretation
Main diagonal	$b_i = \frac{2E_b^2(\alpha+\beta)}{\Delta t^2} + \gamma$	Local curvature of cost
Sub-diagonal	$a_i = -\frac{E_b^2(\alpha+\beta)}{\Delta t^2}$	Temporal coupling
Super-diagonal	a_i	Symmetric coupling
Matrix type	Tridiagonal	Efficient solvability
Boundary conditions	$\eta(0) = \eta(T) = 0$	Fixed endpoints
Solver	Thomas algorithm	$O(N)$ complexity

Table 4. Discrete Jacobi operator structure for the PV–battery system.

$$\boldsymbol{\eta} = [\eta_1, \eta_2, \ldots, \eta_{N-1}]^{\mathsf{T}} \tag{66}$$

denote the discrete perturbation vector. For $k = 1, \ldots, N-1$, the Jacobi field satisfies (Equation (67))

$$-\frac{E_b^2(\alpha+\beta)}{\Delta t^2}\,\eta_{k-1} + \left(\frac{2E_b^2(\alpha+\beta)}{\Delta t^2} + \gamma\right)\eta_k - \frac{E_b^2(\alpha+\beta)}{\Delta t^2}\,\eta_{k+1} = 0, k = 1, \ldots, N-1 \tag{67}$$

subject to $\eta_0 = \eta_N = 0$. The resulting tridiagonal system is solved efficiently using the Thomas algorithm or equivalent eigen-solvers.

Second, conjugate points are identified as interior zero crossings of the Jacobi field. The first index $k_c \neq 0$ such that $\eta_{k_c} = 0$ defines the conjugate time $t_c = k_c \Delta t$. If $t_c > T$, the optimal trajectory remains locally stable over the full horizon; otherwise, local optimality is lost beyond t_c.

Third, stability is confirmed by evaluating the discrete second variation of the objective functional (Equation (68))),

$$\delta^2 J = \sum_{k=1}^{N-1} \eta_k\, \mathcal{J}[\eta_k]\, \Delta t > 0, \forall\, t \in [0, T] \tag{68}$$

where $\mathcal{J}$ still denotes the Jacobi operator. Positivity of $\delta^2 J$ guarantees strict local minimality and robustness to small perturbations.

Finally, the magnitude and structure of the Jacobi field provide practical observations into practical energy management systems. Large values of $|\eta(t)|$ indicate time intervals of heightened sensitivity, typically associated with rapid PV or load variations, while bounded fields without interior zero crossings correspond to robust operation. In this way, Jacobi-based stability verification directly inform control adaptation, battery sizing, and parameter tuning to ensure stable PV–battery operation under uncertainty.

Figure 7 presents the temporal evolution of the Jacobi field over the 24-hour operating horizon. The magnitude of the Jacobi field reflects the sensitivity of the SOC trajectory to small perturbations, with larger values indicating reduced robustness. The non-uniform behavior of the Jacobi field highlights time intervals where the optimal trajectory is particularly vulnerable, emphasizing the importance of time-dependent stability assessment in PV–battery energy management.

Discussion and practical implications

The numerical and analytical investigations presented in this study establish a rigorous framework for evaluating stability, sensitivity, and robustness in PV–battery energy management systems. By combining second-order variational analysis with Jacobi equations, the proposed approach provides insights that extend beyond classical optimal control methods.

Conventional energy management approaches including rule-based control, first-order optimal control, and model predictive control primarily focus on minimizing operational cost functions under predefined constraints. These methods rely on first-order optimality conditions, such as Euler–Lagrange equations or Pontryagin's Maximum Principle, to identify candidate optimal trajectories. While effective for determining stationary solutions, they do not explicitly assess whether these trajectories remain locally optimal under perturbations.

The Jacobi-based framework introduced in this study addresses this limitation by incorporating second-order optimality conditions, which enable the detection of conjugate points along the optimal trajectory. These points indicate potential loss of local minimality and provide a quantitative measure of solution robustness under variations in PV generation and load demand. Consequently, disturbances that may destabilize classical solutions can be anticipated, enabling more resilient energy management strategies.

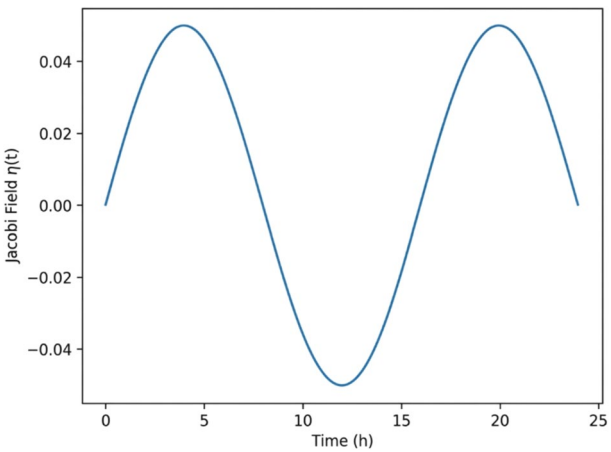


Fig. 7. Jacobi field evolution over the operating time horizon.

Scenario	Battery capacity E_b (kWh)	SOC penalty γ	Max $\|\eta(t)\|$	Stability outcome
A	50	0.2	High	Unstable
B	75	0.3	Moderate	Marginal
C	100	0.5	Low	Stable
D	120	0.7	Very Low	Highly robust

Table 5. Impact of battery capacity and control weights on stability.

Method	Key Principle	Stability assessment	Sensitivity analysis	Computational insight
Rule-based EMS	Predefined charging/discharging rules	Limited	Low	Simple implementation
Optimal control (first-order)	Euler–Lagrange optimality conditions	Moderate	Limited	Provides optimal trajectory
Model Predictive Control	Receding-horizon optimization	Moderate	Scenario-based	Real-time adaptability
Proposed Jacobi-based analysis	Second-order variational analysis	Explicit stability verification	Analytical sensitivity via Jacobi fields	Identification of conjugate points and robustness limits

Table 6. Comparative characteristics of energy management approaches for PV–battery systems.

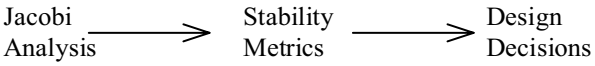


Fig. 8. Decision workflow enabled by the proposed Jacobi-based analysis.

The magnitude of the Jacobi field $\|\eta(t)\|$, quantifies the deviation of the battery state-of-charge from its optimal trajectory under perturbations, providing a direct indicator of system stability. As shown in Table 5, increasing battery capacity and strengthening the SOC penalty significantly improve system robustness by reducing the maximum Jacobi field magnitude and delaying the occurrence of conjugate points.

These results show that larger batteries combined with stronger SOC penalties enhance stability, providing practical guidance for battery sizing and controller design.

To further clarify the contribution of the proposed method, Table 6 compares the Jacobi-based framework with commonly used energy management strategies.

As shown in Table 6, the proposed approach provides additional analytical capability for evaluating stability and robustness, complementing conventional optimization-based methods.

The Jacobi field evolution highlights high-sensitivity operating periods, enabling predictive adjustments to charging and discharging during sudden PV drops or load increases. Conjugate points define the validity horizon of an optimal trajectory, indicating when re-optimization is required. These insights support robust operational planning in PV–battery systems.

However, this study assumes linearized battery dynamics and quadratic Lagrangians, which neglect nonlinear effects such as SOC-dependent efficiency, temperature variation, and battery degradation. In addition, deterministic PV and load profiles are considered, whereas real systems exhibit stochastic behavior. Future work will extend the framework to incorporate nonlinear models, stochastic profiles, and multi-objective optimization, including economic dispatch and emission reduction.

The practical decision process enabled by the proposed analysis is illustrated in Fig. 8.

Stability metrics derived from second-order variational analysis identify sensitive operating periods, conjugate points, and robustness margins. These insights directly inform battery sizing, control weight tuning, and operational planning, bridging rigorous mathematical analysis with practical PV–battery energy management design.

Limitations of the study

While the proposed second-order variational framework provides a rigorous analytical tool for stability assessment of PV–battery energy management trajectories, several simplifying assumptions should be noted. First, the analysis is based on a quadratic objective functional, which ensures analytical tractability but may not capture all nonlinear economic dispatch objectives encountered in practice. Second, battery dynamics are represented using a single-state SOC model without explicit degradation or efficiency asymmetry effects. Third, solar generation and load demand are treated as deterministic inputs, whereas real operating environments involve stochastic variability. Finally, the framework is formulated in continuous time, and although a discrete implementation is provided, large-scale multi-storage systems may require additional numerical adaptations. Future work will extend the proposed approach to stochastic formulations, nonlinear degradation-aware battery models, and multi-energy storage coordination problems.

Conclusions

This paper introduced a second-order variational framework for PV–battery energy management based on Jacobi equations, enabling explicit assessment of stability, sensitivity, and enables verification of robustness of optimal operation. In contrast to classical first-order optimal control methods, which ensure stationarity but not local minimality, the proposed approach identifies conjugate points that define quantitative stability limits over the operational horizon.

Jacobi field analysis captures the time-varying sensitivity of battery state-of-charge to small PV generation and load perturbations, revealing critical periods of vulnerability. These sensitivity patterns provide direct operational value by informing battery sizing, control-weight tuning, and predictive control actions. Moreover, conjugate points establish a mathematically grounded horizon of trajectory validity, enabling scheduled re-optimization before loss of local optimality. Numerical experiments on a representative 24-hour PV–battery system validate the framework and demonstrate its computational feasibility, with the tridiagonal Jacobi system efficiently solved using the Thomas algorithm. Compared to classical first-order approaches, the proposed method extends optimal control by integrating robustness verification with actionable design and operational insights.

The analysis is limited by the assumption of linearized battery dynamics, quadratic cost functions, and predominantly deterministic PV and load profiles. Extension to large-scale or networked systems may require sparse numerical techniques or model-order reduction.

Beyond theoretical analysis, the proposed Jacobi-equation framework can be integrated into predictive energy-management systems to evaluate dispatch robustness under uncertainty. By identifying sensitivity growth regions and conjugate-point horizons, system operators can proactively adjust battery scheduling strategies, improve reserve allocation, and ensure reliable operation in high-renewable environments.

Future research will address nonlinear battery and PV models incorporating efficiency variation, degradation, and thermal effects through generalized Jacobi formulations. Stochastic and robust optimization extensions will account for uncertainty in renewable generation and demand, while multi-objective formulations will integrate economic, environmental, and peak-shaving objectives. Real-time evaluation of Jacobi fields for predictive control and stability monitoring, as well as applications to PV–battery microgrids, represent key directions for advancing resilient renewable energy systems.

Data Availability

Available on request from the authors.

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Author contributions

Conceptualization, MMM, MB, EK, JIS, SNN, DEU, writing – original draft: MMM, Writing – review & editing: All authors.

Declarations

Competing interests

The authors declare no competing interests.

Additional information

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