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Heat transfer optimization in MHD Casson ternary hybrid nanofluid flow using CPC fractional modeling

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Abstract

Efficient thermal management in microchannel systems is essential for modern cooling devices, biomedical systems, energy units, and compact heat exchangers. In this study, the unsteady heat transfer and flow behavior of a Casson-type ternary hybrid nanofluid in a parallel-plate microchannel are investigated under generalized magnetohydrodynamic (MHD) effects. Water is used as the base fluid, while gold (*Au*), copper (*Cu*), and silver (*Ag*) nanoparticles are uniformly suspended to improve the thermal transport capacity of the fluid. The main motivation of this work is to examine how memory effects, magnetic field strength, and enhanced nanoparticle properties influence the velocity and temperature distributions in a non-Newtonian microchannel flow. To describe the memory and hereditary characteristics of the flow, the governing equations are modeled using the Constant Proportional Caputo (CPC) fractional derivative. An implicit finite-difference scheme is also developed to support thus study results numerically. The model is validated by comparing the present profiles with the previous study results, and a close agreement is observed. In addition, a magnitude-based sensitivity analysis is carried out to identify the most influential physical parameters. The results show that the thermal Grashof number has the strongest effect on the velocity response, while the effective Prandtl number is the dominant parameter controlling the temperature field. The CPC fractional model provides smoother and more flexible transient behavior compared with the classical integer-order model. Furthermore, the ternary hybrid nanofluid shows better heat transfer performance than simple nanofluid and hybrid nanofluid cases because of its improved effective thermal conductivity.

Keywords Finite difference, Fractional solution, Casson fluid, Non-Newtonian fluid, Nanofluid, Heat transfer

1 Introduction

The influence of mass and heat transfer is a crucial element of the non-Newtonian fluid flow in some intricate scenarios. Using mass and heat transfer systems to model non-Newtonian fluid flow has many practical uses, such as the preservation of food, the generation of heat, the design of automotive carburetors, etc. In industrial engineering,



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non-Newtonian fluids play a crucial role. Fluid mechanics theorists give them a lot of attention, and to address the engineering significance, a number of theoretical models are developed and subsequently solved. Yang and Du [1] offer a novel investigation of non-Newtonian fluids that includes the flow of the boundary layer and heat transfer analysis. Using A fractional derivative of CE, Ahmad et al. [2] investigated water-based Maxwell hybrid nanofluids using mathematical modeling. Ayub et al. [3] used the effects of magnetic force and Newtonian heating to evaluate heat transmission in viscous flow inside a rotating domain.

Nanofluids are prepared by dispersing very small solid nanoparticles in a base fluid to improve its heat-transfer ability. The presence of nanoparticles usually increases the thermal conductivity of the working fluid, making nanofluids useful in many engineering and biomedical systems. Their applications include drug delivery, solar water heating, engine cooling, electronic cooling, transformer oil cooling, nuclear reactor cooling, and drag-reduction systems. The term “nanofluid” was introduced by Choi et al. [4] during the development of advanced coolants at Argonne National Laboratory. In general, these fluids contain nanoparticles with sizes smaller than 100 nm, which help enhance the thermal performance of conventional fluids. A clear understanding of fluid motion and heat transport is essential for designing efficient thermal systems. With the progress of nanotechnology, researchers have combined ordinary base fluids with different nanoparticles to obtain improved thermal properties. Among these advanced fluids, ternary hybrid nanofluids have attracted considerable attention because they contain three different types of nanoparticles suspended in a single base fluid. This combination can provide better thermal performance than mono and hybrid nanofluids due to the combined effects of the selected nanoparticles. Therefore, ternary hybrid nanofluids are considered promising working fluids for heat-transfer enhancement in modern engineering applications. Several recent studies have explored the behavior of nanofluids and hybrid nanofluids under different physical conditions. Alharbi [5] studied heat transfer in ternary nanofluid flow over a stretching or shrinking sheet and examined the effects of slip conditions and nanoparticle concentration. Usman et al. [6] analyzed three-dimensional power-law nanofluid flow with nonlinear thermal radiation and a uniform magnetic field, showing that these effects strongly influence boundary-layer behavior. Rashdi et al. [7] developed a numerical model for steady two-dimensional Maxwell nanofluid flow and reported that the velocity decreases when the Maxwell parameter increases. Sadighi et al. [8] investigated mass transfer in magnetohydrodynamic power-law nanofluid flow and found that a higher concentration exponent improves mass-transfer behavior.

A significant progress in the research of heat transfer (HT) engineering and nanotechnology, tri-hybrid nanofluids (THNFs) are a step beyond hybrid nanofluids (HNFs). Three types of nanoparticles are masterfully combined with a base fluid in THNFs to form a fantastic blend that enables remarkable enhancement in convective and thermal conductivity heat transfer. By utilizing the peculiarities of three different nanoparticles, THNFs open a new series of opportunities to enhance the effectiveness of heat transfer (HT) in different industrial and technical practices to a significant extent. Alqawasmi et al. [9] explored a 2D time independent model to numerically model the magnetohydrodynamic (MHD) Ternary hybrid nanofluid (THNF) flow. On increasing the estimates of the magnetic parameters, they found out that, with the increase of temperature, there is a decrease in velocity. A Shear-thinning liquid is known as a Casson fluid that has

Infinite viscosity with no shear rate. Infinite viscosity and zero shear rate characterize Casson fluid, a shear-thinning liquid. If the imposed yield stress is greater than the shear stress, the fluid acts as a solid; if it is less than the shear stress, it starts to flow [10]. Casson fluids include things like jelly, intense fruit liquids, soup, tomato sauce, and honey. Human blood can also be used to make Casson fluid [11]. Casson fluid flow across the thin sheet that shrinks with varying thermal radiation and wall temperature, Bhattacharyya et al. [12] found the precise thermal boundary layer solution. Riaz et al. [13] studied nanoparticle solidification in MHD nanofluid flow using non-local fractional kernels and showed how fractional operators help describe heat-transfer behavior in porous media. Ali et al. [14] used a Prabhakar fractional approach to analyze MHD Oldroyd-B fluid flow with slip and Newtonian heating, highlighting the role of memory effects in thermal transport. Ali et al. [15] explored the influence of viscoelasticity on a thermal system in annular cylindrical regions under an inclined magnetic field. Further, recent studies regarding nanofluids can be analyzed in [16–19].

The researchers in the majority of previous studies mainly examined the steady state and seldom included the temporal component. The fluid speed at the surface of the solid must be greater than zero for slip to occur, which causes motion of the fluid with relation to the solid surface. Particularly, the no-slip assumption in microfluidics may lead to inaccurate estimations of flow dynamics; the slip condition has a significant impact on fluid dynamics. Slip may affect heat and mass transfer rates, change the transport properties of the fluid, and result in changes to velocity profiles. To build and improve microfluidic devices, including Microscale heat exchangers and lab-on-a-chip technologies, it is essential to understand slip at the boundary. The interfacial slip boundary condition was employed in Devi et al. [20] to accurately solve the pair stress fluid flow issue via a polluted fluid sphere. Yasin et al. [21] investigated Maxwell fluid with MHD viscosity and elasticity in slip conditions with a concentration on peristalsis. Theoretical and statistical analysis of Navier–Stokes equations with power law slip boundary condition was conducted by Djoko et al. [22]. Saqib et al. [23] analyzed the Casson nanofluid using the AB-fractional derivatives. Examining the unstable MHD flow of a Casson ternary nanofluid over an infinite vertical flat plate is what makes this work unique. In contrast to earlier research that used the Laplace transforms for analytical answers, such as [24] and [25]. The finite difference numerical approach is used in this study to get more precise and reliable results. Numerical computations are performed using the finite difference method. To compare analytical and numerical findings, an analysis is carried out. The performance of significant factors like temperature and velocity is the fundamental objective of the graphical analysis.

Based on the above discussion, it is clear that recent studies have made important progress in the analysis of Casson fluids, MHD flows, nanofluids, hybrid nanofluids, and fractional transport models. However, most available works are mainly limited to simple nanofluids or binary hybrid nanofluids, while the combined effects of ternary hybrid nanoparticles, Casson rheology, microchannel geometry, magnetic field, and fractional memory behavior have not been fully explored. In particular, very limited attention has been given to water-based Au–Cu–Ag ternary hybrid Casson nanofluid flow between parallel plates under a CPC fractional framework. This gap is critical as most of the microchannel flows are sensitive to wall effects, magnetic resistance, composition and the memory dependent transport behavior of the nanoparticles. This motivates

the present study to investigate the unsteady MHD flow and heat transfer of a Casson ternary hybrid nanofluid in a parallel-plate microchannel. The governing equations are derived based on the CPC fractional derivative, which is able to more realistically represent the memory and non-local transport effects than the classical integer-order model. In addition, an implicit finite-difference scheme is formulated to check the validity and to calculate the numerical results. Besides, the classical and fractional solutions are compared, performance of NF, HNF, and THNF cases are assessed, and magnitude-based sensitivity analysis is carried out to find the most influential physical parameters. The validity of the model is also verified by comparing the results obtained in the present work with the published previous work.

2 Mathematical formulation

In this study, we consider the magnetohydrodynamic (MHD) transport of a generalized Casson fluid through a vertical microchannel formed by two infinite parallel plates separated by a distance d . The fluid motion is assumed to occur along the x -direction, while the y -axis is taken normal to the flow direction. Initially, at $t = 0$, the fluid is stationary, and both plates are kept at the same reference temperature T_0 . For $t > 0$, the plate at $y = d$ begins to oscillate suddenly, and its temperature increases from T_0 to T_w . This temperature difference produces a buoyancy effect, which drives the fluid motion and influences both the velocity and thermal boundary layers. The non-Newtonian nature of the Casson fluid, commonly used to model yield-stress materials such as blood, paints, and suspensions, is described through its constitutive relation.

$$\tau_{ij} = \begin{cases} 2 \left(\mu_{thnf} + \frac{P_y}{\sqrt{2\pi}} \right) e_{ij}, \pi > \pi_c \\ 2 \left(\mu_{thnf} + \frac{P_y}{\sqrt{2\pi_c}} \right) e_{ij}, \pi < \pi_c \end{cases} \quad (1)$$

where

$$\gamma = \frac{\mu_{thnf} \sqrt{2\pi_c}}{P_y}. \quad (2)$$

This constitutive relation links the shear stress and strain rate, accounting for the nonlinear characteristics of Casson-type ternary hybrid nanofluids, and is expressed as given in [23, 26]. The governing equations for momentum and energy transport, along with the associated initial and boundary conditions, are then posed for this configuration.

$$\begin{aligned} \rho_{thnf} \frac{\partial u(y, t)}{\partial t} &= \mu_{thnf} \left(1 + \frac{1}{\gamma} \right) \frac{\partial^2 u(y, t)}{\partial y^2} \\ &- \left(\sigma_{thnf} B_0^2 \sin(\theta) + \left(1 + \frac{1}{\gamma} \right) \frac{\mu_{thnf} \varphi_p}{k} \right) u(y, t) + g(\rho \beta_T)_{thnf} (T(y, t) - T_0) \end{aligned} \quad (3)$$

$$(\rho c_p)_{thnf} \frac{\partial T(y, t)}{\partial t} = k_{thnf} \frac{\partial^2 T(y, t)}{\partial y^2} - \frac{\partial q_r}{\partial y} \quad (4)$$

Equations (3) and (4) represent the momentum and energy dynamics, which control the Casson-type ternary hybrid nanofluid flow, respectively. Associated with the following boundary and initial physical conditions:

$$u(y, 0) = 0, T(y, 0) = T_0, \forall y \geq 0 \quad (5)$$

$$u(0, t) = 0, T(0, t) = T_0$$

$$u(d, t) = 0, T(d, t) = T_w, t > 0 \quad (6)$$

where u and v are the velocity components of the ternary hybrid nanofluids, γ denotes the Casson fluid parameter, temperature of the fluid is denoted by T , $\bar{g}$ is the acceleration due to gravity, B_0 is the external magnetic field, result of the actual deformation rate component is φ_p , c_p is the specific heat at constant pressure, T_0 is the uniform reference temperature, and k is the permeability of the porous material, respectively. In addition, the other mathematical symbols such as μ_{thnf} , ρ_{thnf} , k_{thnf} , σ_{thnf} , $(\rho c_p)_{thnf}$, and $(\rho \beta_T)_{thnf}$ correspond the viscosity, density, thermal conductivity, electrical conductivity, specific heat capacitance, and thermal expansion coefficient of the ternary hybrid nanofluid, respectively. The expression for the radiative heat flux is given as follows:

$$q_r = -\frac{4\sigma_1}{3k_\epsilon} \frac{\partial T^4}{\partial y} \quad (7)$$

T^4 is expended by T_0 using Taylor series as

$$T^4 \cong 4T_0^3 T - 3T_0^4 \quad (8)$$

By substituting Eqs. (7) and (8) into Eq. (4), the energy equation is obtained as:

$$(\rho c_p)_{thnf} \frac{\partial T(y, t)}{\partial t} = k_{thnf} \left(1 + \frac{16\sigma_1 T_0^3}{3k_{thnf} k_\epsilon} \right) \frac{\partial^2 T(y, t)}{\partial y^2} \quad (9)$$

The governing equations are converted into nondimensional form by introducing suitable dimensionless variables into the original dimensional model. In this process, the physical quantities are scaled with their corresponding characteristic values, such as reference length, velocity, time, and temperature. This scaling removes the units from the equations and reduces the number of controlling parameters. The momentum and energy equations then become simpler and more compact, and can be solved analytically and are more convenient for examining the influence of various physical parameters.

$$A_o \frac{\partial w(y, t)}{\partial t} = A_1 \left(1 + \frac{1}{\gamma} \right) \frac{\partial^2 w(y, t)}{\partial y^2} - \left(A_2 M \sin(\theta) + \left(1 + \frac{1}{\gamma} \right) \frac{A_1}{K} \right) w(y, t) + A_3 Gr T(y, t) \quad (10)$$

$$A_4 \frac{\partial T(y, t)}{\partial t} = \frac{1}{Pr_{eff}} \frac{\partial^2 T(y, t)}{\partial y^2} \quad (11)$$

With corresponding initial and boundary conditions:

$$w(y, 0) = 0, T(y, 0) = 0, \quad (12)$$

$$w(0, t) = 0, T(0, t) = 0,$$

$$w(d, t) = 0, T(d, t) = 1. \quad (13)$$

where

$$\begin{aligned}
Pr &= \frac{\nu_f(\rho C_p)_f}{k_f}, Rd = \frac{16\sigma_1 T_0^3}{3k_f k_{thnf}}, M = \frac{\sigma_f B_0^2 d^2}{\rho_f \nu_f} \\
\frac{1}{K} &= \frac{\varphi_p d^2}{k}, Gr = \frac{g\beta_f d^3 \Delta T}{\nu_f^2}, Pr_{eff} = \frac{Pr}{A_5 + Rd} \\
A_0 &= (1 - \phi_{Cu}) \left[(1 - \phi_{Au}) \left((1 - \phi_{Ag}) + \phi_{Ag} \frac{\rho_{Ag}}{\rho_f} \right) + \phi_{Au} \frac{\rho_{Au}}{\rho_f} \right] + \phi_{Cu} \frac{\rho_{Cu}}{\rho_f}, \\
A_1 &= [(1 - \phi_{Cu})(1 - \phi_{Au})(1 - \phi_{Ag})]^{2.5}, A_2 = \frac{\sigma_{thnf}}{\sigma_f}, \\
A_3 &= (1 - \phi_{Cu}) \left[(1 - \phi_{Au}) \left\{ (1 - \phi_{Ag}) + \phi_{Ag} \frac{(\rho\beta_T)_{Ag}}{(\rho\beta_T)_f} \right\} + \phi_{Au} \frac{(\rho\beta_T)_{Au}}{(\rho\beta_T)_f} \right] + \phi_{Cu} \frac{(\rho\beta_T)_{Cu}}{(\rho\beta_T)_f}, \\
A_4 &= (1 - \phi_{Cu}) \left[(1 - \phi_{Au}) \left\{ (1 - \phi_{Ag}) + \phi_{Ag} \frac{(\rho C_p)_{Ag}}{(\rho C_p)_f} \right\} + \phi_{Au} \frac{(\rho C_p)_{Au}}{(\rho C_p)_f} \right] + \phi_{Cu} \frac{(\rho C_p)_{Cu}}{(\rho C_p)_f}, \\
A_5 &= \frac{k_{thnf}}{k_f}.
\end{aligned}$$

The physical experimental data of the nanoparticles and their correlations of the ternary hybrid nanofluids are given in Tables 1 and 2, respectively.

2.1 Constant proportional caputo (CPC)

The concepts of proportionality and classical differentiation are combined in the CPC operator, a fractional calculus operator. This operator not only can model specific phenomena that cannot be modeled using typical integer order operators, but it also provides a more versatile structure for modeling dynamic systems.

The Constant Proportional Caputo (CPC) derivative is a generalized fractional derivative that extends the classical Caputo derivative by including a constant proportional term. This operator is useful for describing systems where the present behavior depends not only on the current state but also on the past history of the process. In fractional fluid dynamics, the CPC derivative is commonly used to represent memory and hereditary effects in complex transport phenomena, such as heat transfer and nanofluid flow. For a sufficiently smooth function, the CPC fractional derivative of order α can be written as follows:

$${}_0^{CPC} D_t^\alpha f(t) = \frac{M(\alpha)}{1 - \alpha} \int_0^t f'(\tau) e^{-\frac{\alpha}{1-\alpha}(t-\tau)} d\tau$$

where, $M(\alpha)$ and $0 < \alpha < 1$ is a function of normalization.

Table 1 Physical aspects of the base fluid and nanomaterials [27, 28]

Properties	(H_2O)	Ag	Au	Cu
$\rho(Kg/m^3)$	997.1	10,500	19,300	8933
$c_p(J/KgK)$	4179	235	129	385
$k(W/mK)$	0.613	429	318	401
$\sigma(S/m)$	0.05	6.30×10^7	4.10×10^7	5.96×10^7
$\beta(1/K) \times 10^{-5}$	21	1.89	1.4	1.67

Table 2 Thermophysical properties of the ternary hybrid nanofluids

Property	Hybrid nanofluid (HNF)	Ternary hybrid nanofluid (THNF)
Viscosity	$\mu_{hnf} = \mu_f (1 - \phi_{Au})^{-2.5} (1 - \phi_{Ag})^{-2.5}$	$\frac{\mu_{tnf}}{\mu_f} = \frac{1}{(1 - \phi_{Cu})^{2.5} (1 - \phi_{Au})^{2.5} (1 - \phi_{Ag})^{2.5}}$
Density	$\rho_{hnf} = \rho_f \left[(1 - \phi_{Au}) + \phi_{Au} \left(\frac{\rho_{Au}}{\rho_f} \right) \right] (1 - \phi_{Ag}) + \phi_{Ag} \rho_{Ag}$	$\frac{\rho_{tnf}}{\rho_f} = (1 - \phi_{Cu}) \left[(1 - \phi_{Au}) \left((1 - \phi_{Ag}) + \phi_{Ag} \frac{\rho_{Ag}}{\rho_f} \right) + \phi_{Au} \frac{\rho_{Au}}{\rho_f} \right] + \phi_{Cu} \frac{\rho_{Cu}}{\rho_f}$
Heat capacity	$(\rho C_P)_{hnf} = (\rho C_P)_f \left[(1 - \phi_{Ag}) (1 - \phi_{Au}) + \phi_{Au} \left(\frac{(\rho C_P)_{Au}}{(\rho C_P)_f} \right) \right] + \phi_{Ag} \left(\frac{(\rho C_P)_{Ag}}{(\rho C_P)_f} \right)$	$\frac{(\rho C_P)_{tnf}}{(\rho C_P)_f} = \phi_{Cu} \frac{(\rho C_P)_{Cu}}{(\rho C_P)_f} + (1 - \phi_{Cu}) \left[(1 - \phi_{Au}) \left((1 - \phi_{Ag}) + \phi_{Ag} \frac{(\rho C_P)_{Ag}}{(\rho C_P)_f} \right) + \phi_{Au} \frac{(\rho C_P)_{Au}}{(\rho C_P)_f} \right]$
Thermal conductivity	$\frac{k_{hnf}}{k_f} = \frac{k_{Ag} + (m-1)k_{nf} - (m-1)\phi_{Ag}(k_{nf} - k_{Ag})}{k_{Ag} + (m-1)k_{nf} + \phi_{Ag}(k_{nf} - k_{Ag})}$ Step1 : basefluid + Ag $\frac{k_{Au} + (m-1)k_f - (m-1)\phi_{Au}(k_f - k_{Au})}{k_{Au} + (m-1)k_f + \phi_{Au}(k_f - k_{Au})}$ Step2 : addAu	$\frac{k_{tnf}}{k_{nf}} = \frac{k_{Ag} + 2k_{hnf} - 2\phi_{Ag}(k_{hnf} - k_{Ag})}{k_{Ag} + 2k_{hnf} + \phi_{Ag}(k_{hnf} - k_{Ag})}$ $\frac{k_{hnf}}{k_{nf}} = \frac{k_{Au} + 2k_{nf} - 2\phi_{Au}(k_{nf} - k_{Au})}{k_{Au} + 2k_{nf} + \phi_{Au}(k_{nf} - k_{Au})}$ $\frac{k_{nf}}{k_f} = \frac{k_{Cu} + 2k_f - 2\phi_{Cu}(k_f - k_{Cu})}{k_{Cu} + 2k_f + \phi_{Cu}(k_f - k_{Cu})}$
Electrical conductivity	$\frac{\sigma_{hnf}}{\sigma_f} = \frac{\sigma_{Ag} + (m-1)\sigma_{nf} - (m-1)\phi_{Ag}(\sigma_{nf} - \sigma_{Ag})}{\sigma_{Ag} + (m-1)\sigma_{nf} + \phi_{Ag}(\sigma_{nf} - \sigma_{Ag})}$ Step1 : basefluid + Ag $\frac{\sigma_{Au} + (m-1)\sigma_f - (m-1)\phi_{Au}(\sigma_f - \sigma_{Au})}{\sigma_{Au} + (m-1)\sigma_f + \phi_{Au}(\sigma_f - \sigma_{Au})}$ Step2 : addAu	$\frac{\sigma_{tnf}}{\sigma_{nf}} = \frac{\sigma_{Ag} + 2\sigma_{hnf} - 2\phi_{Ag}(\sigma_{hnf} - \sigma_{Ag})}{\sigma_{Ag} + 2\sigma_{hnf} + \phi_{Ag}(\sigma_{hnf} - \sigma_{Ag})}$ $\frac{\sigma_{hnf}}{\sigma_{nf}} = \frac{\sigma_{Au} + 2\sigma_{nf} - 2\phi_{Au}(\sigma_{nf} - \sigma_{Au})}{\sigma_{Au} + 2\sigma_{nf} + \phi_{Au}(\sigma_{nf} - \sigma_{Au})}$ $\frac{\sigma_{nf}}{\sigma_f} = \frac{\sigma_{Cu} + 2\sigma_f - 2\phi_{Cu}(\sigma_f - \sigma_{Cu})}{\sigma_{Cu} + 2\sigma_f + \phi_{Cu}(\sigma_f - \sigma_{Cu})}$

The Laplace transformation of the Constant Proportional Caputo(CPC) operator can be described as:

$$\mathcal{L} [{}_0^{CPC} D_t^\alpha f(\tau)] = \left\{ \frac{k_1(\alpha)}{q} + k_0(\alpha) \right\} q^\alpha f^{(s)}(\tau) \quad (14)$$

2.2 Laplace transform

Laplace transformation is an integral transformation that can be employed to transform a time dependent function into a complex-frequency-dependent function. It makes it simple because it converts the differential equations into algebraic equations in the field.

$$\mathcal{L} \{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt, s \in \mathbb{C}$$

Since this method converts the complex integral operators to algebraic representations, this method is especially useful in solving fractional-order derivative equations that involve CPC derivatives.

3 Solution of the problem

In this section, both fractional analytical and numerical solutions of the proposed model are developed. First, the Laplace transform is applied to the fractional Casson-type model to obtain exact analytical solutions. This transformation converts fractional derivatives into algebraic expressions in the Laplace domain, making the governing equations easier to solve. In addition, the FDM is used to compute the numerical solution of the governing equations. This enables the comparison of analytical and numerical results for validation and accuracy.

3.1 Fractional solution

In the Laplace domain the governing system can be simplified into an algebraic form by transforming Eq. (10) and (11) by use of the fractional Laplace transformation, and use of Eq. (14) along with the boundary conditions implied by the prescription of the problem in (12). This change provides a systematic way of resolving the model and eases the control of fractional derivatives in the time domain.

3.1.1 Solution of temperature field

A clear way to determine the temperature field is to use the Constant Proportional Caputo (CPC) fractional derivative together with the Laplace transform. For this purpose, the CPC operator is applied to the governing Eq. (11). Then, using the Laplace transform definition given in Eq. (14) and applying the modified boundary conditions, the final analytical expression for the temperature distribution is obtained.

$$\frac{\partial^2 \bar{T}(y, q)}{\partial y^2} - A_4 Pr_{eff} \{k_1(\alpha) + qk_0(\alpha)\} q^{\alpha-1} \bar{T}(y, q) = 0 \quad (15)$$

The corresponding boundary conditions can be obtained as follows by applying the Laplace transformation:

$$\bar{T}(0, q) = 0, \bar{T}(1, q) = \frac{1}{q}$$

The following solutions are obtained by using the transformed boundary conditions mentioned above.

$$\bar{T}(y, q) = \frac{1}{q} \frac{\sinh \left(y \sqrt{A_4 Pr_{eff} \left\{ \frac{k_1(\alpha)}{q} + k_0(\alpha) \right\} q^\alpha} \right)}{\sinh \left(\sqrt{A_4 Pr_{eff} \left\{ \frac{k_1(\alpha)}{q} + k_0(\alpha) \right\} q^\alpha} \right)} \quad (16)$$

For the Laplace inverse of the above solution, write the above expression in its standard series form as written below.

$$\bar{T}(y, q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(A_4 Pr_{eff})^{\frac{n}{2}}}{n!} \binom{n/2}{j} k_1(\alpha)^{\frac{n}{2}-j} k_0(\alpha)^j [-((2m+1)(1+y)) + ((2m+1)(1-y))] q^{j+\frac{n}{2}(\alpha-1)-}$$

Now taking its Laplace inverse, we get

$$T(y, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(A_4 Pr_{eff})^{\frac{n}{2}}}{n!} \binom{n/2}{j} k_1(\alpha)^{\frac{n}{2}-j} \frac{[-((2m+1)(1-y)) - ((2m+1)(1-y))] }{\Gamma(\rho_{n,j})} k_0(\alpha)^j t^{\rho_{n,j}-1}$$

where $\rho_{n,j} = 1 - j - \frac{n}{2}(\alpha - 1)$.

3.1.2 Solution for the velocity field

By taking the Laplace transform of the velocity Eq. (10), the time-dependent partial differential equation is converted into an ordinary differential equation with respect to the spatial variable. In this process, the time derivatives are replaced by algebraic expressions in the Laplace domain. Therefore, the transformed velocity equation can be written in a simpler and more compact form as follows

$$A_o \bar{w}(y, q) = A_1 \left(1 + \frac{1}{\gamma} \right) \frac{\partial^2 \bar{w}(y, q)}{\partial y^2} - \left(A_2 M \sin(\theta) + \left(1 + \frac{1}{\gamma} \right) \frac{A_1}{K} \right) \bar{w}(y, q) + A_3 Gr \bar{T}(y, q) \quad (17)$$

However, when the boundaries are transformed using the Laplace transformation, they become:

$$\bar{w}(0, q) = 0; \bar{w}(1, q) = 0$$

The momentum field can be obtained in a systematic way by using the Laplace transform. For this purpose, the transformed boundary conditions are applied together with the temperature solution already derived in Eq. (16). This allows the velocity solution to be determined in a consistent and simplified form.

$$\bar{w}(y, q) = \left(\frac{A_3 Gr}{A_1 \left(1 + \frac{1}{\gamma} \right) q \left(A_4 Pr_{eff} \left\{ \frac{k_1(\alpha)}{q} + k_0(\alpha) \right\} q^\alpha \right) - \left(\frac{1}{A_1 \left(1 + \frac{1}{\gamma} \right)} \left(A_2 M \sin(\theta) + \left(1 + \frac{1}{\gamma} \right) \frac{A_1}{K} + A_0 q \right) \right)} \frac{1}{\sinh \left(\sqrt{\frac{1}{A_1 \left(1 + \frac{1}{\gamma} \right)} \left(A_2 M \sin(\theta) + \left(1 + \frac{1}{\gamma} \right) \frac{A_1}{K} + A_0 q \right)} \right)} \right)$$

$$\sinh \left(y \sqrt{\frac{1}{A_1 \left(1 + \frac{1}{\gamma}\right)} \left(A_2 M \sin(\theta) + \left(1 + \frac{1}{\gamma}\right) \frac{A_1}{K} + A_0 q \right)} \right) - \frac{A_3 G_r}{A_1 \left(1 + \frac{1}{\gamma}\right) q \left(A_4 P r_{eff} \left\{ \frac{h_0(\alpha)}{\gamma} + k_0(\alpha) \right\} q^n \right) - \left(\frac{1}{A_1 \left(1 + \frac{1}{\gamma}\right)} \left(A_2 M \sin(\theta) + \left(1 + \frac{1}{\gamma}\right) \frac{A_1}{K} + A_0 q \right) \right) \sinh \left(y \sqrt{A_4 P r_{eff} \left\{ \frac{h_0(\alpha)}{\gamma} + k_0(\alpha) \right\} q^n} \right)} \quad (18)$$

Stehfest and Tzou's techniques will be used to obtain the Laplace inverse of the velocity solution mentioned above.

3.2 Numerical solution

In this section, the finite difference method is used to solve the non-dimensional governing equations numerically. This method is suitable for the present unsteady and coupled tri-hybrid nanofluid flow problem because it provides a stable and accurate numerical framework. The spatial domain is divided into a uniform grid, while the solution is advanced in time using a fixed time step. To preserve second-order accuracy, the diffusion terms are approximated at the half-time level, and the source and coupling terms are treated in a consistent manner. At each time level, the discretized equations produce tridiagonal linear systems, which are efficiently solved by the Thomas algorithm. The initial condition represents a fluid initially at rest, while the boundary conditions are applied at the wall and in the far-field region. The previously developed non-dimensional model is written as follows:

$$A_o \frac{\partial w(y, t)}{\partial t} = A_1 \left(1 + \frac{1}{\gamma} \right) \frac{\partial^2 w(y, t)}{\partial y^2} - \left(A_2 M \sin(\theta) + \left(1 + \frac{1}{\gamma} \right) \frac{A_1}{K} \right) w(y, t) + A_3 G_r T(y, t) \quad (19)$$

$$A_4 \frac{\partial T(y, t)}{\partial t} = \frac{1}{Pr_{eff}} \frac{\partial^2 T(y, t)}{\partial y^2} \quad (20)$$

We discretize the spatial domain $[0, L]$, with a uniform grid. Let Δx be the spacing between adjacent points and N be the total number of grid points, including the boundaries at $x=0$ and $x=L$. These quantities are related by.

$$\Delta x = \frac{L}{N-1}, N = \frac{L}{\Delta x} + 1; x_i = i\Delta x, i = 0, 1, 2, 3, \dots, N-1.$$

so $x_0 = 0$, and $x_{N-1} = L$. In this case, L denotes the domain's physical length, Δx determines the mesh's resolution, and N is the number of nodes that the numerical scheme, such as Finite Difference, uses. Consider the explicit FDM, non-dimensional discretized temperature equation as follows.

$$A_4 \left(\frac{T_i^{n+1} - T_i^n}{\Delta t} \right) = \frac{1}{Pr_{eff}} \left(\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta y^2} \right)$$

or in the more simplified form as:

$$T_i^{n+1} = (1 - 2\gamma) T_i^n + \gamma T_{i+1}^n + \gamma T_{i-1}^n$$

It clearly demonstrates that only its two closest neighbors at the new time level and the same three nodes at the prior time level are connected to the unknown value T_i^{n+1} . While the Dirichlet boundary data imposes a ramp temperature at the wall, the initial condition T_0^i sets a quiescent start everywhere.

$$T_i^0 = 0 \forall i, T_0^n = 0, T_{N-1}^n = 1$$

Which can be written in matrix form as follows:

$$T^{n+1} = \left(1 + \gamma \tilde{L}\right) T^n + b_T^{n+1} \quad (21)$$

where

$$\gamma = \frac{A_4 \Delta t}{Pr_{eff} \Delta y^2}, L = \frac{1}{\Delta y^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix}, \tilde{L} = \Delta y^2 L,$$

$$b_T^n = \begin{bmatrix} \gamma T_0^n \\ 0 \\ 0 \\ \vdots \\ \gamma T_{N-1}^n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \gamma \end{bmatrix}$$

In matrix notation, the internal unknowns get combined into a vector T^n . The discrete Laplacian (second difference) on inner nodes is represented by the matrix L , which is symmetric, sparse, and strictly diagonally dominating.

Similarly, the finite difference technique can be used to numerically discretize the velocity field.

$$A_o \left(\frac{w_i^{n+1} - w_i^n}{\Delta t} \right) = A_1 \left(1 + \frac{1}{\gamma} \right) \left(\frac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{\Delta y^2} \right) - \left(A_2 M \sin(\theta) + \left(1 + \frac{1}{\gamma} \right) \frac{A_1}{K} \right) w_i^n + A_3 G_r T_i^n$$

$$A_o \frac{\partial w(y, t)}{\partial t} = A_1 \left(1 + \frac{1}{\gamma} \right) \frac{\partial^2 w(y, t)}{\partial y^2} - \left(A_2 M \sin(\theta) + \left(1 + \frac{1}{\gamma} \right) \frac{A_1}{K} \right) w(y, t) + A_3 Gr T(y, t)$$

is the second-order accurate discretization of a linear parabolic balance for the velocity. The thermal buoyancy enters additively through the step T^{n+1} . The discretized equation mentioned above can be expressed more simply as follows:

$$w_i^{n+1} = \left[1 - 2\lambda - \Delta t \left(A_2 M \sin(\theta) + \left(1 + \frac{1}{\gamma} \right) \frac{A_1}{K} \right) \right] w_i^n + \lambda (w_{i-1}^n + w_{i+1}^n) + \Delta t A_3 G_r T_i^n$$

Which can be written in the simplified matrix form as

$$w^{n+1} = \left[\left(1 - \Delta t \left(A_2 M \sin(\theta) + \left(1 + \frac{1}{\gamma} \right) \frac{A_1}{K} \right) \right) I + \lambda D \right] w^n + \Delta t A_3 G_r T^n + b_w^{n+1} \quad (22)$$

where:

$$\lambda = \frac{\Delta t}{\gamma_0 \Delta y^2}, L = \frac{1}{\Delta y^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix}, \text{ and } D = (\Delta y^2) L$$

The proposed scheme results in linear systems of tridiagonal form that can be efficiently solved at each time step with a linear time algorithm. It correctly handles the diffusion terms and uses the time-dependent Dirichlet boundary conditions by using consistent half-step boundary values. Because the temperature field is firstly calculated separately and then applied as a known half-step source term in the momentum equation, the

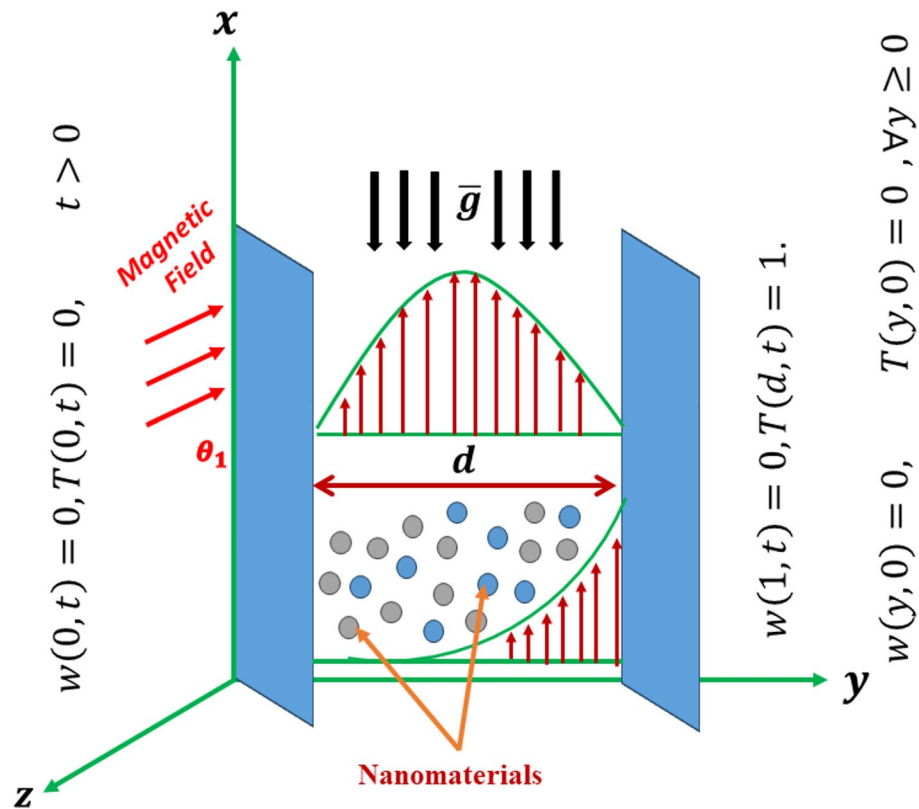


Fig. 1 Flow diagram for vertical microchannel

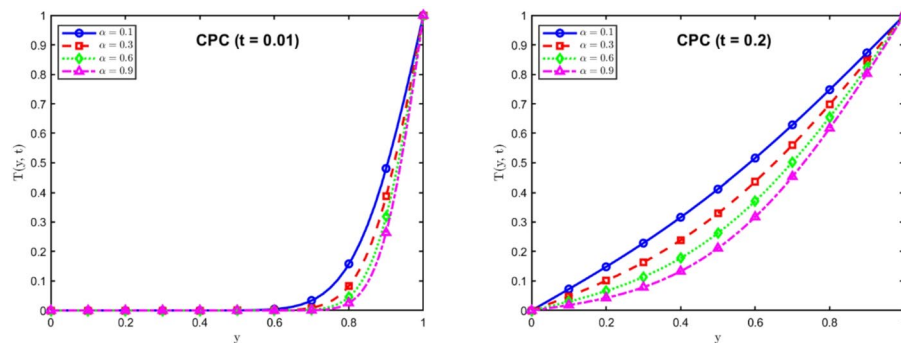


Fig. 2 Impact of fractional constraints (α) on the temperature field at different time values

coupling between temperature and velocity is simple and stable. The accuracy of the method can be examined by carrying out mesh and spatial refinement analysis and the reliability of the method can be tested by comparing the numerical results with the analytical results provided by the Laplace transform.

4 Results discussion

In this study, the unsteady flow of Casson-type nanofluid over a vertical plate in the presence of an external magnetic field has been examined. The influence of the ramped and constant wall temperature conditions on the thermal behaviour of the fluid is given special consideration. The CPC fractional derivative is added to the mathematical model to capture the memory effects and non-classical transport behavior better than the

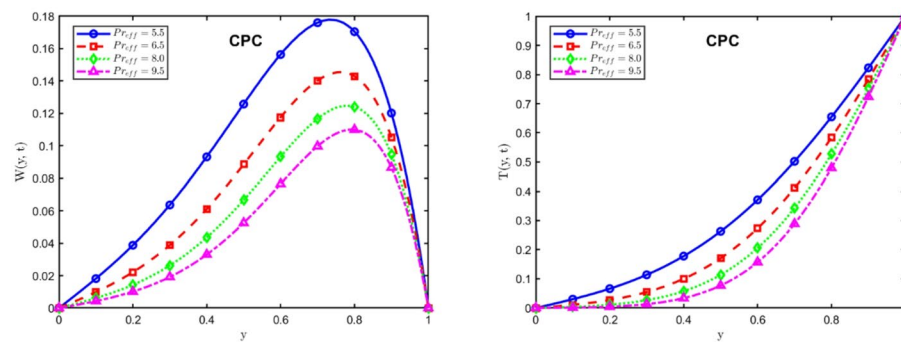


Fig. 3 Impact of Pr_{eff} on the velocity and temperature field

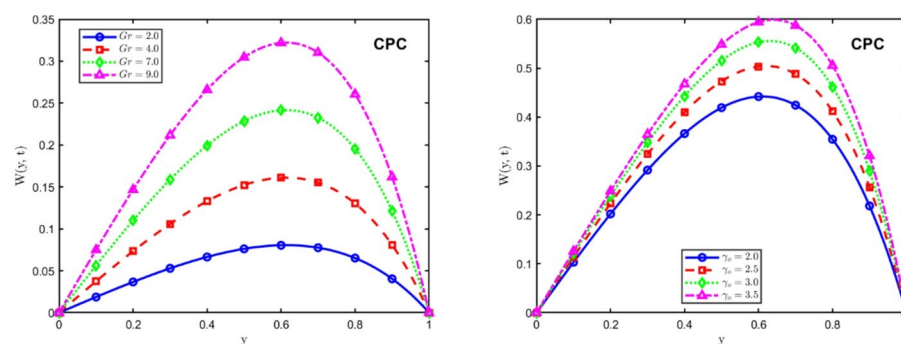


Fig. 4 Impact of Gr and γ_o on the velocity field

classical approach. Gold (Au), Copper (Cu), and Silver (Ag) nanoparticles are suspended in the base fluid (water), to improve the thermal conductivity and heat transfer performance. The physical configuration of the problem is shown in Fig. 1, and the variations in the velocity and temperature fields with the effect of different parameters are shown in Figs. 2–9. Further, the heat and mass transfer performances are quantitatively assessed by the skin friction coefficient, Nusselt and Sherwood number, which are tabulated and compared to the performance under different boundary conditions.

4.1 Impact of parameters by fractional solution

The effect of the fractional parameter (α) of the CPC on the temperature profile in the flow region is illustrated in Fig. 2. It is noted that an increase in α causes a decrease in temperature. Such behavior is closely associated with the memory effect and the non-locality of the fractional derivative, which slows down the process of heat diffusion and hence the thickness of a thermal boundary layer.

The influence of the Pr_{eff} on the velocity and temperature distribution is shown in Fig. 3. The profiles both decrease as it increases. Both profiles get decreases as it gets larger. This is because the higher the ratio of Pr_{eff} the lower the diffusivity of heat in the fluid as compared to the diffusivity of momentum, and hence less heat transfer takes place within the fluid. Both the velocity and the thermal fields decrease regularly.

As Gr value is increased, the momentum profile also increases, as seen in Fig. 4. This behavior is mainly due to the strengthening of the buoyancy force. The buoyancy effect due to temperature variation gets stronger as Gr increases, and the fluid motion is accelerated by this, resulting in higher velocity profile. The effects of the Casson resistance

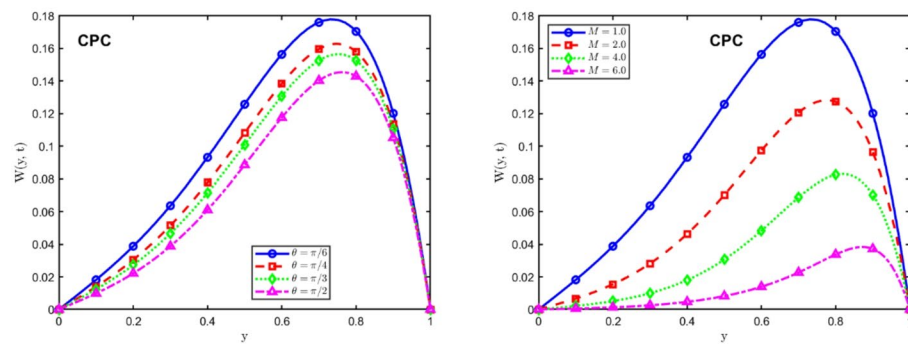


Fig. 5 Impact of θ and M on the velocity field

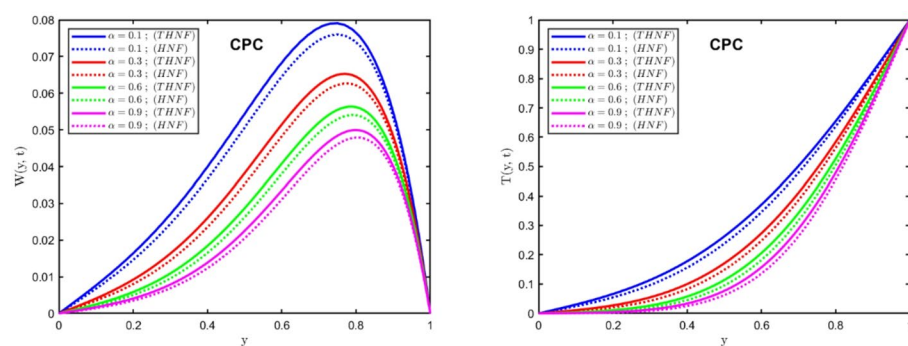


Fig. 6 Comparison of THNF and HNF by varying the value of α

parameter (γ_0) on the velocity field are also shown in Fig. 4. The addition of (γ_0) changes the effective resistance of the fluid, and in the present case, increases the velocity distribution. Therefore, the effect of both Gr and (γ_0) is considerable and significant on the momentum transport of the Casson nanofluid.

Figure 5 shows the effects of the magnetic parameter (M) and the inclination angle (θ) on the streamwise velocity using the fractional solution. As the magnetic parameter increases, the Lorentz force becomes stronger. This force acts as a resistance against the fluid motion and therefore reduces the velocity within the boundary layer. The influence of the magnetic field also depends on the angle at which it is applied. A stronger retarding effect is observed when the magnetic field is applied perpendicular to the flow direction, where the Lorentz force reaches its maximum effect.

Figure 6 shows the comparison between HNF and THNF for the momentum and temperature fields at different values of the α . It is observed that the THNF produces a stronger effect than the HNF in both velocity and temperature distributions. This improvement is mainly due to the higher effective thermal conductivity and better thermophysical properties of the ternary nanoparticle mixture. The presence of three nanoparticles enhances the heat transfer capacity of the base fluid more effectively than the two-particle hybrid suspension. As a result, the THNF shows improved thermal transport and a more noticeable influence on the flow behavior.

4.2 Magnitude-based sensitivity analysis

Figure 7 presents the magnitude-based sensitivity analysis for both HNF and THNF. The left plot shows the sensitivity of the $|S_p^{Cf}|$, while the right plot represents the $|S_p^{Nu}|$

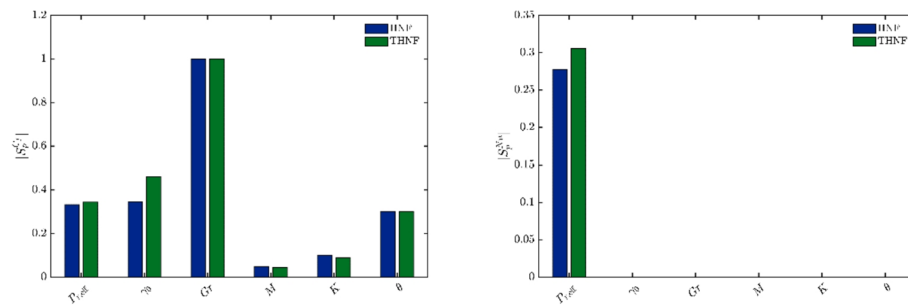


Fig. 7 Magnitude-based sensitivity analysis for both HNF and THNF fluids

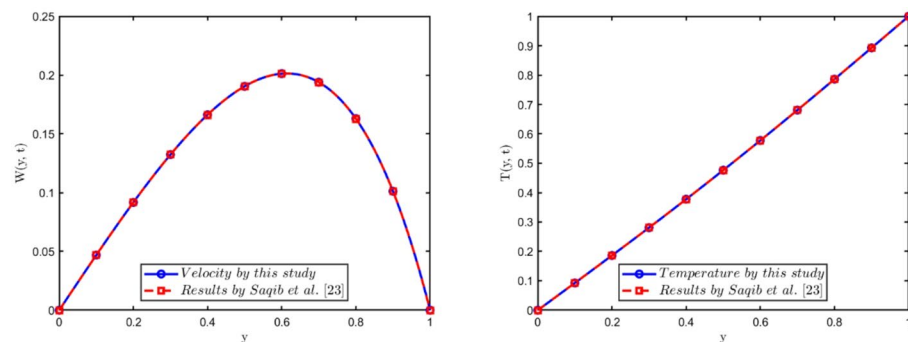


Fig. 8 Comparison of this study with the results of Saqib et al. [23]

response. It is observed that the thermal Grashof number (Gr) has the highest sensitivity in the $|S_p^{Cf}|$ case. In the $|S_p^{Nu}|$ case, the effective Prandtl number (Pr_{eff}) is the only parameter showing a clear influence. This occurs because the temperature equation is directly controlled by Pr_{eff} , while the other parameters, such as Gr , M , K , θ , and γ_o , mainly belong to the momentum equation and do not directly affect the temperature distribution in the present formulation. Therefore, their sensitivity values in the thermal response are almost zero. The comparison between HNF and THNF also shows that THNF generally provides a stronger sensitivity response than HNF. This indicates that the ternary hybrid nanofluid has a more pronounced effect on the transport characteristics due to its improved effective thermophysical properties.

For each controlling parameter p , the normalized local sensitivity of a physical quantity Y is defined as

$$S_p^Y = \frac{p}{Y} \frac{\partial Y}{\partial p}$$

To compare the relative strength of different parameters, the absolute magnitude of the sensitivity coefficient is used:

$$|S_p^Y| = \frac{p_o}{Y_o} \left[\frac{Y(p_o + \Delta p) - Y(p_o - \Delta p)}{2\Delta p} \right]$$

Here, p_o is the baseline value of the selected parameter, Y_o is the corresponding baseline value of the output quantity, and Δp is a small perturbation in the parameter. In this study, the perturbation is taken as $\Delta p = 0.05p_o$ with 5% variation.

Table 4 The numerical comparison of all the Nusselt and Sherwood numbers

Pr_{eff}	Nusselt	Sherwood
6	0.19447004	0.388130808
6.5	0.157623479	0.356646889
7	0.127380934	0.329076031
7.5	0.102679228	0.304850267
8	0.08258482	0.283485561
8.5	0.066294274	0.26457163
9.5	0.042508965	0.232762676
10	0.033967394	0.219327266
10.5	0.027109018	0.207246117

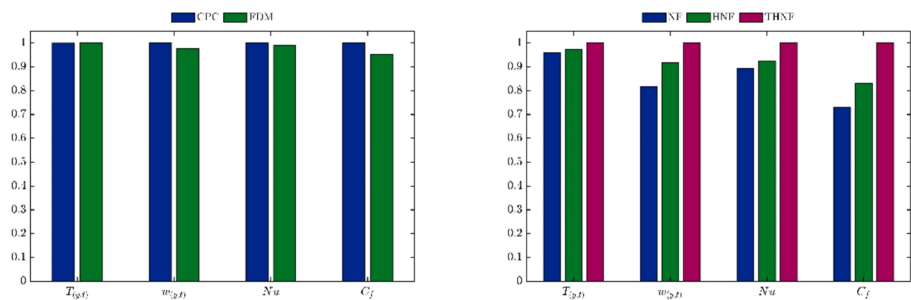


Fig. 9 Comparison of fractional and numerical solutions with different hybrid nanofluids

4.3 Validation of this study

To validate the present results, the velocity and temperature profiles are compared with the previously reported results of Saqib et al. [23] in Fig. 8. The comparison shows a very close agreement between both solutions, which confirms the correctness and reliability of the present analytical and numerical approach. The matching behavior of the curves also indicates that the applied boundary conditions and solution procedure are properly implemented. Furthermore, Table 4 presents a numerical comparison of the Nusselt number and Sherwood number for different values of the effective Prandtl number (Pr_{eff}). It is observed that both quantities decrease as Pr_{eff} increases. This decreasing trend shows that higher Pr_{eff} reduces thermal diffusion in the fluid, which weakens the heat transfer rate near the wall.

4.4 Comparison of fractional and numerical solution

Finally, a quantitative and graphical comparison between the fractional and numerical solutions is presented. Figure 9 shows the comparison between the CPC fractional solution (at $\alpha = 1.0$) and the FDM solution. A close agreement between both approaches can be observed, which confirms the reliability of the numerical scheme. However, the CPC-based fractional solution gives a more flexible and physically meaningful description because it includes memory effects, which are usually absent in the classical FDM approach. The comparison also shows that the THNF has a stronger influence on the flow and thermal fields compared with the HNF and simple nanofluid NF. This improvement is mainly related to the enhanced effective thermal conductivity and better thermophysical properties produced by the combined suspension of three different nanoparticles. Furthermore, Table 5 presents the numerical values obtained from both CPC and FDM solutions.

Table 5 The numerical comparison of the fractional and numerical solutions

Pr_{eff}	Temp CPC	Temp FDM	Vel CPC	Vel FDM
1	0.113979866	0.115784947	0.20030128	0.199866029
1.5	0.052826026	0.054013566	0.190437699	0.189480991
2	0.025359526	0.026095832	0.177534568	0.176170258
2.5	0.012431657	0.012873723	0.164202859	0.162580002
3.5	0.00310824	0.003259226	0.140172706	0.138332864
4.5	0.000804821	0.000854369	0.120793825	0.118937781
5.5	0.000213647	0.000229565	0.105408873	0.103615332
6.5	5.78729E-05	6.29295E-05	0.093080915	0.091378114
7.5	1.59602E-05	1.75583E-05	0.083056632	0.081451883

5 Conclusion

In this study, the unsteady flow of a Casson nanofluid in a vertical microchannel under MHD effects has been investigated. The channel is filled with water-based nanofluid containing Ag, Au, and Cu nanoparticles to improve the thermal performance of the fluid. The CPC fractional derivative is used to include memory effects and non-classical diffusion behavior in the model. In addition, a second-order implicit finite difference method is applied to compute numerical results and support the analytical findings. The main conclusions of the present study are given below:

- The CPC fractional model gives a more flexible description of the flow and heat transfer process due to the presence of memory effects not taken into account in the classical model.
- As the fractional parameter tends to $\alpha \rightarrow 1$, the present fractional model is converted to the classical Casson nanofluid model. This is to ensure consistency of the proposed formulation.
- The temperature profile decreases as Pr_{eff} increases. This is due to the fact that higher Pr_{eff} decreases the thermal diffusivity and makes it less possible for heat to penetrate the fluid.
- The velocity field is higher with the Gr . The buoyancy force is stronger with greater Gr , and this speeds up the fluid motion.
- The velocity profile is decreased by the magnetic parameter M . This is due to the fact that the Lorentz force represents a counter-inertial force opposing the fluid flow.
- The Casson-related parameter γ_o plays an important role in controlling the momentum transport. This variation alters the effective resistance of the fluid and thus the velocity profile.
- Ternary hybrid nanofluid is more thermally and flow performance than hybrid nanofluid and simple nanofluid.
- A sensitivity analysis based on the magnitude is also carried out to determine the most significant parameters. It is found that the dominant effect on the velocity response is the result of Gr , whereas for the thermal response, the dominant parameter is Pr_{eff} .
- For the validation of the present model, the velocity and temperature profile obtained are compared with the Saqib et al. [23] results. The good agreement of both results establishes the accuracy and reliability of the present solution method.

The overall results of the fractional and numerical solutions are reasonably well matched and help to confirm the accuracy of the formulation and computational scheme. The

results demonstrate the applicability of the CPC fractional modeling to complex nano-fluid flow problems with a significant role of memory and non-local transport effects.

From an engineering point of view, the present results can be used in the design of microchannel heat exchangers, cooling systems, biomedical flow devices, and MHD-based thermal systems. The improved heat transfer behavior of ternary hybrid nano-fluids can also support advanced thermal management applications in energy and electronic cooling technologies.

Acknowledgements

The authors extend their appreciation to the Deanship of Scientific Research, Islamic University of Madinah, Saudi Arabia, for funding this research work.

Author contributions

****M.S., H.R.:** Conceptualization, Methodology, Software, Formal analysis, Validation; Writing – original draft. ****J.N.:** Writing–original draft, Data curation, Investigation, Visualization, Validation. ****U.K.:** Conceptualization, Writing–original draft, Writing–review & editing, Supervision, Resources. ****M.D.S; S.M.H.:** Validation, Investigation, Writing–review & editing, Formal analysis; Project administration; Funding acquisition, software.

Funding

The authors received support in the form of funding from the Deanship of Scientific Research, Islamic University of Madinah, Saudi Arabia, for funding this research work.

Data Availability

The data that support the findings of this study are available from the corresponding author (Umair Khan) upon reasonable request.

Declarations

Ethical approval and Consent to participate

Not applicable.

Consent to publish

Not applicable.

Competing Interests

The authors declare no competing interests.

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Received: 10 March 2026 / Accepted: 15 June 2026

Published online: 26 June 2026

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